CONTROL AND STABILIZATION OF THE BENJAMIN-ONO EQUATION ON A PERIODIC DOMAIN

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Abstract. It was proved by Linares and Ortega in [24] that the *linearized* Benjamin-Ono equation posed on a periodic domain $\mathbb{T}$ with a distributed control supported on an arbitrary subdomain is exactly controllable and exponentially stabilizable. The aim of this paper is to extend those results to the *full* Benjamin-Ono equation. A feedback law in the form of a localized damping is incorporated in the equation. A smoothing effect established with the aid of a propagation of regularity property is used to prove the semi-global stabilization in $L^2(\mathbb{T})$ of weak solutions obtained by the method of vanishing viscosity. The local well-posedness and the local exponential stability in $H^s(\mathbb{T})$ are also established for $s > 1/2$ by using the contraction mapping theorem. Finally, the local exact controllability is derived in $H^s(\mathbb{T})$ for $s > 1/2$ by combining the above feedback law with some open loop control.

1. Introduction

The Benjamin-Ono (BO) equation can we written as

$$u_t + \mathcal{H}u_{xx} + uu_x = 0,$$

where $u = u(x,t)$ denotes a real-valued function of the variables $x \in \mathbb{R}$ and $t \in \mathbb{R}$, and $\mathcal{H}$ denotes the Hilbert transform defined as

$$\widehat{\mathcal{H}u}(\xi) = -i\,\mathrm{sgn}(\xi)\,\hat{u}(\xi).$$

This integro-differential equation models the propagation of internal waves in stratified fluids of great depth (see [4, 33]) and turns out to be important in other physical situations as well (see [9, 18, 26]). Among noticeable properties of this equation we can mention that: (i) it defines a Hamiltonian system; (ii) it admits infinitely many conserved quantities (see [6]); (iii) it can be solved by an analogue of the inverse scattering method (see [2]); (iv) it admits (multi)soliton solutions (see [6]).

In this paper, we consider the BO equation posed on the periodic domain $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$:

$$u_t + \mathcal{H}u_{xx} + uu_x = 0, \quad x \in \mathbb{T},\ t \in \mathbb{R}, \tag{1.1}$$

where the Hilbert transform $\mathcal{H}$ is defined now by

$$\widehat{(\mathcal{H}u)}_k = -i\,\mathrm{sgn}(k)\hat{u}_k.$$

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The two first conserved quantities are

$$I_1(t) = \int_{\mathbb{T}} u(x,t)dx$$

and

$$I_2(t) = \int_{\mathbb{T}} u^2(x,t)dx.$$

From the historical origins [4, 33] of the BO equation, involving the behavior of stratified fluids, it is natural to think I_1 and I_2 as expressing conservation of volume (or mass) and energy, respectively.

The Cauchy problem for the equation (1.1) in the real line has been intensively studied for many years ([45, 17, 1, 32, 31, 20, 19, 46, 5, 16, 29, 13, 14]). In the periodic case, there have been several recent developments. (See for instance [28, 30, 29] and the references therein.) The best known result so far [28, 29] is that the Cauchy problem is well-posed in the space

$$H^s_0(\mathbb{T}) = \{u \in H^s(\mathbb{T});\ \hat{u}_0 := \frac{1}{2\pi}\int_{\mathbb{T}} u(x)\, dx = 0\}$$

for $s \geq 0$. Moreover, the corresponding solution map $(u_0 \to u)$ is real analytic from the space $H^0_0(\mathbb{T})$ to the space $C([0,T], H^0_0(\mathbb{T}))$.

In this paper we will study the equation (1.1) from a control point of view with a forcing term $f = f(x,t)$ added to the equation as a control input:

$$u_t + \mathcal{H}u_{xx} + uu_x = f(x,t), \quad x \in \mathbb{T},\ t \in \mathbb{R}, \tag{1.2}$$

where f is assumed to be supported in a given open set $\omega \subset \mathbb{T}$. The following exact control problem and stabilization problem are fundamental in control theory.

Exact Control Problem: Given an initial state u_0 and a terminal state u_1 in a certain space, can one find an appropriate control input f so that the equation (1.2) admits a solution u which satisfies $u(\cdot,0) = u_0$ and $u(\cdot,T) = u_1$?

Stabilization Problem: Can one find a feedback law $f = Ku$ so that the resulting closed-loop system

$$u_t + \mathcal{H}u_{xx} + uu_x = Ku, \quad x \in \mathbb{T},\ t \in \mathbb{R}^+$$

is asymptotically stable as $t \to +\infty$?

Those questions were first investigated by Russell and Zhang in [44] for the Korteweg-de Vries equation, which serves as a model for propagation of surface waves along a channel:

$$u_t + u_{xxx} + uu_x = f, \quad x \in \mathbb{T},\ t \in \mathbb{R}. \tag{1.3}$$

In their work, in order to keep the *mass* $I_1(t)$ conserved, the control input is chosen to be of the form

$$f(x,t) = (Gh)(x,t) := a(x)\left(h(x,t) - \int_{\mathbb{T}} a(y)h(y,t)\, dy\right)$$

where h is considered as a new control input, and $a(x)$ is a given nonnegative smooth function such that $\{x \in \mathbb{T};\ a(x) > 0\} = \omega$ and

$$2\pi[a] = \int_{\mathbb{T}} a(x)\, dx = 1.$$

For the chosen a, it is easy to see that

$$\frac{d}{dt}\int_{\mathbb{T}} u(x,t)\,dx = \int_{\mathbb{T}} f(x,t)dx = 0 \quad \forall t\in\mathbb{R}$$

for any solution $u=u(x,t)$ of the system

$$u_t + u_{xxx} + uu_x = Gh, \quad x\in\mathbb{T},\ t\in\mathbb{R}. \tag{1.4}$$

Thus the *mass* of the system is indeed conserved.

The control of dispersive nonlinear waves equations on a periodic domain has been extensively studied in the last decade: see e.g. [44, 40, 23] for the Korteweg-de Vries equation, [27] for the Boussinesq system, [42] for the BBM equation, and [11, 38, 21, 41, 22] for the nonlinear Schrödinger equation. By contrast, the control theory of the BO equation is at its early stage. The following results are due to Linares and Ortega [24].

Theorem A. [24] *Let $s\ge 0$ and $T>0$ be given. Then for any $u_0,u_1\in H^s(\mathbb{T})$ with $[u_0]=[u_1]$ one can find a control input $h\in L^2(0,T,H^s(\mathbb{T}))$ such that the solution of the system*

$$u_t + \mathcal{H}u_{xx} = Gh, \qquad u(x,0)=u_0(x) \tag{1.5}$$

satisfies $u(x,T)=u_1(x)$.

In order to stabilize (1.5), Linares and Ortega employed a simple control law

$$h(x,t) = -G^*u(x,t).$$

The resulting closed-loop system reads

$$u_t + \mathcal{H}u_{xx} = -GG^*u.$$

Theorem B. [24] *Let $s\ge 0$ be given. Then there exist some constants $C>0$ and $\lambda>0$ such that for any $u_0\in H^s(\mathbb{T})$, the solution of*

$$u_t + \mathcal{H}u_{xx} = -GG^*u, \qquad u(x,0)=u_0(x)$$

satisfies

$$\|u(\cdot,t)-[u_0]\|_{H^s(\mathbb{T})} \le Ce^{-\lambda t}\|u_0-[u_0]\|_{H^s(\mathbb{T})} \qquad \forall t\ge 0.$$

The extension of those results to the full BO equation (1.4) turns out to be a very hard task. Indeed, it is by now well known that the contraction principle cannot be used to establish the local well-posedness of BO in $H^s_0(\mathbb{T})$ for $s\ge 0$. The method of proof in [28, 29] used strongly Tao's gauge transform, and it is not clear whether this approach can be followed when an additional control term is present in the equation.

For the sake of simplicity, we shall assume from now on that $[u_0]=0$, so that $u(t)$ has a zero mean value for all times.

To stabilize the BO equation, we consider the following feedback law

$$f = -G(D(Gu))$$

where $\widehat{Du}_k = |k|\hat{u}_k$. Scaling in (1.3) by u gives (at least formally)

$$\frac12\|u(T)\|^2_{L^2(\mathbb{T})} + \int_0^T \|D^{\frac12}(Gu)\|^2_{L^2(\mathbb{T})}dt = \frac12\|u_0\|^2_{L^2(\mathbb{T})}. \tag{1.6}$$

This suggests that the energy is dissipated over time. On the other hand, (1.6) reveals a smoothing effect, at least in the region $\{a > 0\}$. Using a *propagation of regularity property* in the same vein as in [11, 21, 22, 23], we shall prove that the smoothing effect holds everywhere, i.e.

$$\|u\|_{L^2(0,T;H^{\frac{1}{2}}(\mathbb{T}))} \leq C(T, \|u_0\|). \tag{1.7}$$

Using this smoothing effect and the classical compactness/uniqueness argument, we shall first prove that the corresponding closed-loop equation is semi-globally exponentially stable.

Theorem 1.1. *Let $R > 0$ be given. Then there exist some constants $C = C(R)$ and $\lambda = \lambda(R)$ such that for any $u_0 \in H^0_0(\mathbb{T})$ with $\|u_0\| \leq R$, the weak solutions in the sense of vanishing viscosity of*

$$u_t + \mathcal{H}u_{xx} + uu_x = -GDGu, \qquad u(x,0) = u_0(x) \tag{1.8}$$

satisfy

$$\|u(t)\| \leq Ce^{-\lambda t}\|u_0\| \qquad \forall t \geq 0.$$

A weak solution of (1.8) in the sense of vanishing viscosity is a distributional solution of (1.8) $u \in C_w(\mathbb{R}^+, H^0_0(\mathbb{T})) \cap L^2_{loc}(\mathbb{R}^+, H^{\frac{1}{2}}_0(\mathbb{T}))$ that may be obtained as a weak limit in a certain space of solutions of the BO equation with viscosity

$$u_t + (\mathcal{H} - \varepsilon)u_{xx} + uu_x = -GDGu, \qquad u(x,0) = u_0(x) \tag{1.9}$$

as $\varepsilon \to 0^+$ (see below Definition 2.11 for a precise definition). The issue of the *uniqueness* of the weak solutions in the sense of vanishing viscosity seems challenging.

Using again the smoothing effect (1.7), one can extend (at least locally) the exponential stability from $H^0_0(\mathbb{T})$ to $H^s_0(\mathbb{T})$ for $s > 1/2$.

Theorem 1.2. *Let $s \in (\frac{1}{2}, 2]$. Then there exists $\rho > 0$ such that for any $u_0 \in H^s_0(\mathbb{T})$ with $\|u_0\|_{H^s(\mathbb{T})} < \rho$, there exists for all $T > 0$ a unique solution $u(t)$ of (1.8) in the class $C([0,T], H^s_0(\mathbb{T})) \cap L^2(0,T,H^{s+\frac{1}{2}}_0(\mathbb{T}))$. Furthermore, there exist some constants $C > 0$ and $\lambda > 0$ such that*

$$\|u(t)\|_s \leq Ce^{-\lambda t}\|u_0\|_s \qquad \forall t \geq 0.$$

Finally, incorporating the same feedback law $f = -G(D(Gu))$ in the control input to obtain a smoothing effect, one can derive an exact controllability result for the full equation as well.

Theorem 1.3. *Let $s \in (\frac{1}{2}, 2]$ and $T > 0$ be given. Then there exists $\delta > 0$ such that for any $u_0, u_1 \in H^s_0(\mathbb{T})$ satisfying*

$$\|u_0\|_{H^s(\mathbb{T})} \leq \delta, \quad \|u_1\|_{H^s(\mathbb{T})} \leq \delta$$

one can find a control input $h \in L^2(0,T,H^{s-\frac{1}{2}}(\mathbb{T}))$ such that the system (1.4) admits a solution $u \in C([0,T], H^s_0(\mathbb{T})) \cap L^2(0,T,H^{s+\frac{1}{2}}_0(\mathbb{T}))$ satisfying

$$u(x,0) = u_0(x), \quad u(x,T) = u_1(x).$$

Note that it would be desirable to have a control input h in the class $L^2(0,T,H^s(\mathbb{T}))$, but this will require to adapt the analysis in [28, 29]. Note also that a global controllability result in $H^0_0(\mathbb{T})$ would follow from Theorems 1.1 and 1.3 if Theorem 1.3 were also true for $s=0$.

The paper is organized as follows. Section 2 is concerned with the local well-posedness and the stability properties of (1.8). We first prove the global well-posedness of (1.9) in the energy space $H^0_0(\mathbb{T})$, by using classical energy estimates (Theorem 2.1). Next, we establish several technical properties, namely a commutator estimate (Lemma 2.5), a propagation of regularity property (Propositions 2.7 and 2.16), and a unique continuation property (Proposition 2.8) that are used to derive the exponential stability of (1.9) with a decay rate *independent of* ε (Theorem 2.10). This leads to the proofs of Theorems 1.1 and 1.2. Finally, the control properties of (1.4) are investigated in Section 3.

2. Stabilization of BO with a localized damping

2.1. Semi-global exponential stabilization in $L^2(\mathbb{T})$.

Pick any function

$$a\in C^\infty(\mathbb{T},\mathbb{R}^+)\ \text{ with }\ \int_{\mathbb{T}} a(x)dx=1 \tag{2.10}$$

decomposed as $a(x)=\sum_{k\in\mathbb{Z}}\hat{a}_k e^{ikx}$.

We are interested in the stability properties of the BO equation with localized damping

$$u_t+\mathcal{H}u_{xx}+(\frac{u^2}{2})_x=-G(D(Gu)),\qquad u(0)=u_0, \tag{2.11}$$

where

$$\widehat{\mathcal{H}u}_k=-i\,\mathrm{sgn}(k)\hat{u}_k,\quad \widehat{D^s u}_k=|k|^s\hat{u}_k,\quad (Gu)(x)=a(x)(u(x)-\int_{\mathbb{T}}a(y)u(y)dy). \tag{2.12}$$

We shall assume that $u_0\in H^0_0(\mathbb{T})$, where for any $s\in\mathbb{R}$,

$$H^s_0(\mathbb{T})=\{u=\sum_{k\in\mathbb{Z}}\hat{u}_k e^{ikx}\in H^s(\mathbb{T});\ \hat{u}_0=0\}.$$

Let $(u,v)=\int_{\mathbb{T}}u(x)v(x)dx$ denote the usual scalar product in $L^2(\mathbb{T})$ with $\|u\|=\|u\|_{L^2(\mathbb{T})}$ as associated norm, and for any $s\in\mathbb{R}$, let $(u,v)_s=((1-\partial_x^2)^{\frac{s}{2}}u,(1-\partial_x^2)^{\frac{s}{2}}v)$ denote the scalar product in $H^s(\mathbb{T})$ with corresponding norm $\|u\|_s=(u,u)_s^{\frac12}$. Let $\langle x\rangle:=(1+|x|^2)^{\frac12}$ for any $x\in\mathbb{R}$.

Note that for $s<0$ and $u\in H^s(\mathbb{T})$, Gu has to be understood as

$$Gu=a\left(u-\langle u,a\rangle_{H^s(\mathbb{T}),H^{-s}(\mathbb{T})}\right).$$

Assuming that $u_0\in H^0_0(\mathbb{T})$, we obtain (formally) by scaling in (2.11) by u that

$$\frac12\|u(T)\|^2+\int_0^T\|D^{\frac12}(Gu)\|^2dt=\frac12\|u_0\|^2. \tag{2.13}$$

This suggests that the energy is dissipated over time. On the other hand, (2.13) reveals a smoothing effect, at least in the region $\{a>0\}$. Using a *propagation of regularity property* in the same vein as in [11, 21, 22, 23], we shall prove that the smoothing effect holds everywhere, i.e.

$$u \in L^2(0,T;H^{\frac{1}{2}}(\mathbb{T})). \tag{2.14}$$

Of course, a rigorous derivation of (2.13) requires enough regularity for u, e.g.

$$u \in L^2(0,T,H^1(\mathbb{T})) \cap C([0,T],H^0_0(\mathbb{T})). \tag{2.15}$$

As there is a gap between (2.14) and (2.15), we are let to put some artificial viscosity in (2.11) (parabolic regularization method) to derive in a rigorous way the energy identity for the $\varepsilon-$BO equation

$$u_t + \mathcal{H}u_{xx} + uu_x = \varepsilon u_{xx} - G(D(Gu)), \qquad u(0) = u_0. \tag{2.16}$$

We shall prove the global well-posedness (GWP) of (2.16) in H^0_0, together with the semi-global exponential stability in H^0_0 with a decay rate *uniform* in $\varepsilon > 0$. Letting $\varepsilon \to 0$, this will give the semi-global exponential stability in H^0_0 of the weak solutions $u \in C_w([0,+\infty),H^0_0(\mathbb{T}))$ of (2.11) obtained as limits of the (strong) solutions of (2.16). The (difficult) issue of the uniqueness of a weak solution to (2.11) will not be addressed here.

We first establish the GWP of (2.16).

Theorem 2.1. *Let $\varepsilon > 0$ and $u_0 \in H^0_0(\mathbb{T})$. Then for any $T > 0$ there exists a unique solution $u \in C([0,T],H^0_0(\mathbb{T})) \cap L^2(0,T;H^1(\mathbb{T}))$ of (2.16). Moreover*

$$u \in C((0,T],H^2(\mathbb{T})) \cap C^1((0,T],H^1(\mathbb{T})), \tag{2.17}$$

and for any $t \geq 0$

$$\frac{1}{2}\|u(t)\|^2 + \varepsilon\int_0^t \|u_x(\tau)\|^2 d\tau + \int_0^t \|D^{\frac{1}{2}}(Gu)(\tau)\|^2 d\tau = \frac{1}{2}\|u_0\|^2. \tag{2.18}$$

Proof: The proof of Theorem 2.1 is divided into five parts. Note that the weak smoothing effect (2.14) will be established later, as it is not needed here.

Step 1. Linear Theory

We consider the linear system

$$u_t + (\mathcal{H} - \varepsilon)u_{xx} + G(D(Gu)) = 0, \qquad u(0) = u_0.$$

Let $Au = (\mathcal{H} - \varepsilon)u_{xx}$ with domain $\mathcal{D}(A) = H^2_0(\mathbb{T}) \subset H^0_0(\mathbb{T})$, and $Bu = G(D(Gu))$. Clearly $G \in \mathcal{L}(H^r(\mathbb{T}),H^r_0(\mathbb{T}))$ for all $r \in \mathbb{R}$, hence $B \in \mathcal{L}(H^1_0(\mathbb{T}),H^0_0(\mathbb{T}))$. Let $\theta_0 \in (\arctan\varepsilon^{-1},\pi/2)$. Then, for $\theta_0 < |\arg\lambda| \leq \pi$, we have

$$\|(A-\lambda)^{-1}\| \leq \sup_{k\neq 0}|(\varepsilon + i\,\mathrm{sgn}\; k)k^2 - \lambda|^{-1} \leq \frac{C}{|\lambda|}.$$

It follows that A is a sectorial operator (see [15, Definition 1.3.1]) in $H^0_0(\mathbb{T})$. Note that $\sigma(A) = \{(\varepsilon + i\,\mathrm{sgn}\; k)k^2;\ k \in \mathbb{Z}^*\}$. Therefore, Re $\sigma(A) \geq \varepsilon$ and $A^{-\alpha}$ is meaningful for all $\alpha > 0$. Since for all $s > 0$

$$\|A^{-\frac{s}{2}}u\|^2_{H^s(\mathbb{T})} \leq C\sum_{k\neq 0}|\varepsilon + i\,\mathrm{sgn}\; k|^{-s}|\hat{u}_k|^2 \leq C\|u\|^2_{L^2(\mathbb{T})}$$

we infer that $BA^{-\frac{1}{2}} \in \mathcal{L}(H_0^0(\mathbb{T}))$. It follows from [15, Corollary 1.4.5] that the operator $\mathcal{A} := A + B$ is also sectorial, so that $-\mathcal{A}$ generates an analytic semigroup $\big(\mathcal{S}(t)\big)_{t\geq 0} = (e^{-t\mathcal{A}})_{t\geq 0}$ on $H_0^0(\mathbb{T})$ according to [15, Theorem 1.3.4]. Note that, by [15, Theorem 1.4.8], $D((A+B+\lambda)^\alpha) = D(A^\alpha) = H_0^{2\alpha}(\mathbb{T})$ for all $\alpha \geq 0$ and $\lambda > 0$ large enough, hence

$$\mathcal{S}(t)H_0^s(\mathbb{T}) \subset H_0^s(\mathbb{T}), \qquad \forall t > 0,\ \forall s \geq 0.$$

Let us derive estimates for the solutions of the Cauchy problem

$$u_t + \mathcal{A}u = f, \qquad u(0) = u_0. \tag{2.19}$$

For any $T > 0$ and any $s \in \mathbb{N}$, let

$$Y_{s,T} = C([0,T]; H_0^s(\mathbb{T})) \cap L^2(0,T; H_0^{s+1}(\mathbb{T})) \tag{2.20}$$

be endowed with the norm

$$\|u\|_{Y_{s,T}} = \|u\|_{L^\infty(0,T;H^s(\mathbb{T}))} + \|u\|_{L^2(0,T;H^{s+1}(\mathbb{T}))}. \tag{2.21}$$

Lemma 2.2. *We have for some constant $C_0 = C_0(\varepsilon, s, T)$*

$$\|u\|_{Y_{s,T}} \leq C_0 \left(\|u_0\|_s + \|f\|_{L^1(0,T,H^s(\mathbb{T}))}\right),$$

u denoting the mild solution of (2.19) *associated with* $(u_0, f) \in H_0^s(\mathbb{T}) \times L^1(0,T,H_0^s(\mathbb{T}))$.

Proof of Lemma 2.2. It is well known from classical semigroup theory that

$$\|u\|_{L^\infty(0,T,H^s(\mathbb{T}))} \leq C\left(\|u_0\|_s + \|f\|_{L^1(0,T,H^s(\mathbb{T}))}\right).$$

Next we estimate $\|u\|_{L^2(0,T,H^{s+1}(\mathbb{T}))}$. We first assume $u_0 \in H_0^{s+2}(\mathbb{T})$ and $f \in C([0,T]; H_0^{s+2}(\mathbb{T}))$, so that $u \in C([0,T]; H_0^{s+2}(\mathbb{T})) \cap C^1([0,T]; H_0^s(\mathbb{T}))$. Taking the scalar product of each term of (2.19) by u in $H^s(\mathbb{T})$ results in

$$\frac{1}{2}\|u(t)\|_s^2 + \varepsilon\int_0^t \|u_x\|_s^2\, d\tau + \int_0^t (G(D(Gu)), u)_s\, d\tau = \frac{1}{2}\|u_0\|_s^2 + \int_0^t (f,u)_s\, d\tau. \tag{2.22}$$

The identity (2.22) is also true for $u_0 \in H_0^s(\mathbb{T})$ and $f \in L^1(0,T,H_0^s(\mathbb{T}))$, by density. The following claim is needed.

Claim 1. For any $s \in \mathbb{R}$, there exists a constant $C = C(s) > 0$ such that

$$-\big(G(D(Gu)), u\big)_s \leq C\|u\|_s^2 - \|D^{\frac{1}{2}}(Gu)\|_s^2 \qquad \forall u \in H_0^{s+1}(\mathbb{T}).$$

Proof of Claim 1. We have

$$\begin{aligned}\big(G(D(Gu)), u\big)_s &= \big((1-\partial_x^2)^{\frac{s}{2}} G(D(Gu)), (1-\partial_x^2)^{\frac{s}{2}} u\big)\\ &= \big([(1-\partial_x^2)^{\frac{s}{2}}, G]D(Gu), (1-\partial_x^2)^{\frac{s}{2}} u)\\ &\qquad + (G(1-\partial_x^2)^{\frac{s}{2}} D(Gu), (1-\partial_x^2)^{\frac{s}{2}} u)\\ &=: I_1 + I_2.\end{aligned}$$

Since $a \in C^\infty(\mathbb{T})$, we easily obtain that

$$\|[(1-\partial_x^2)^{\frac{s}{2}}, G]u\| \leq C\|u\|_{s-1}.$$

It follows that

$$|I_1| \leq C\|u\|_s^2.$$

On the other hand

$$\begin{aligned} I_2 &= \big((1-\partial_x^2)^{\frac{s}{2}}D(Gu), G(1-\partial_x^2)^{\frac{s}{2}}u\big) \\ &= \|(1-\partial_x^2)^{\frac{s}{2}}D^{\frac{1}{2}}(Gu)\|^2 + ((1-\partial_x^2)^{\frac{s}{2}}(Gu), D[G,(1-\partial_x^2)^{\frac{s}{2}}]u), \end{aligned}$$

hence

$$-I_2 \le C\|u\|_s^2 - \|D^{\frac{1}{2}}(Gu)\|_s^2.$$

The claim is proved.

Combining Claim 1 with (2.22), we obtain that for $t = T$

$$\begin{aligned} &\frac{1}{2}\|u(T)\|_s^2 + \varepsilon\int_0^T \|u_x(\tau)\|_s^2 d\tau + \int_0^T \|D^{\frac{1}{2}}(Gu)\|_s^2 d\tau \\ &\le \frac{1}{2}\|u_0\|_s^2 + C\|u\|^2_{L^2(0,T,H^s(\mathbb{T}))} + \frac{1}{2}\|u\|^2_{L^\infty(0,T,H^s(\mathbb{T}))} + \frac{1}{2}\|f\|^2_{L^1(0,T,H^s(\mathbb{T}))} \\ &\le C\big(\|u_0\|_s^2 + \|f\|^2_{L^1(0,T,H^s(\mathbb{T}))}\big). \end{aligned}$$

The proof of Lemma 2.2 is achieved. $\square$

Remark 2.3. *We observe that when $u_0 \equiv 0$ in* (2.19) *then*

$$\|\int_0^t \mathcal{S}(t-\tau)f(\tau)\,d\tau\|_{Y_{s,T}} \le C(\epsilon, s, T)\,\|f\|_{L^1(0,T,H^s(\mathbb{T}))}, \tag{2.23}$$

and when $f \equiv 0$ in (2.19)

$$\|\mathcal{S}(t)u_0\|_{Y_{s,T}} \le C(\epsilon, s, T)\|u_0\|_{H^s(\mathbb{T})}. \tag{2.24}$$

Step 2. Local Well-posedness in $H_0^s(\mathbb{T})$, $s \ge 0$

We prove the following

Proposition 2.4. *Let $s \ge 0$. For any $u_0 \in H_0^s(\mathbb{T})$, there exists some $T > 0$ such that the problem* (2.16) *admits a unique solution $u \in Y_{s,T}$.*

Proof. Write (2.16) in its integral form

$$u(t) = \mathcal{S}(t)u_0 - \int_0^t \mathcal{S}(t-\tau)(uu_x)(\tau)d\tau$$

where the spatial variable is suppressed throughout. For given $u_0 \in H_0^s(\mathbb{T})$, let $r > 0$ and $T > 0$ be constants to be determined. Define a map Γ on the closed ball

$$B = \big\{v \in Y_{s,T};\ \|v\|_{Y_{s,T}} \le r\big\}$$

of $Y_{s,T}$ by

$$\Gamma(v)(t) = \mathcal{S}(t)u_0 - \int_0^t \mathcal{S}(t-\tau)(vv_x)(\tau)\,d\tau.$$

We aim to prove that Γ contracts in B for T small enough and r conveniently chosen. To that end, we shall prove the following estimates

$$\|\Gamma(v)\|_{Y_{s,T}} \le C_0\|u_0\|_s + C_1T^{\frac{1}{4}}\|v\|^2_{Y_{s,T}}, \quad \forall v \in B, \tag{2.25}$$

$$\|\Gamma(v^1) - \Gamma(v^2)\|_{Y_{s,T}} \le C_1T^{\frac{1}{4}}(\|v^1\|_{Y_{s,T}} + \|v^2\|_{Y_{s,T}})\|v^1 - v^2\|_{Y_{s,T}} \quad \forall v^1, v^2 \in B. \tag{2.26}$$

From Lemma 2.2 and Remark 2.3, it is adduced that

$$\begin{aligned}\|\Gamma(v^1)-\Gamma(v^2)\|_{Y_{s,T}} &\le C\|v^1v^1_x-v^2v^2_x\|_{L^1(0,T,H^s(\mathbb{T}))}\\ &\le C\int_0^T\big(\|v^1-v^2\|_{L^\infty}\|v^1+v^2\|_{s+1}+\|v^1+v^2\|_{L^\infty}\|v^1-v^2\|_{s+1}\big)d\tau\\ &\le CT^{\frac14}\|v^1-v^2\|_{Y_{s,T}}\big(\|v^1\|_{Y_{s,T}}+\|v^2\|_{Y_{s,T}}\big)\end{aligned}$$

where we used the fact that

$$\int_0^T\|v\|^2_{L^\infty}dt\le C\int_0^T\|v\|_1\|v\|dt\le C\sqrt{T}\|v\|_{L^\infty(0,T,L^2(\mathbb{T}))}\|v\|_{L^2(0,T,H^1(\mathbb{T}))}.$$

This yields (2.26). (2.25) follows from Lemma 2.2, Remark 2.3 and (2.26). Choosing $r>0$ and $T>0$ so that

$$\begin{cases} r=2C_0\|u_0\|_s,\\ 2rC_1T^{\frac14}\le\frac12,\end{cases}\tag{2.27}$$

we obtain that

$$\|\Gamma(v^1)\|_{Y_{s,T}}\le r,\qquad \|\Gamma(v^1)-\Gamma(v^2)\|_{Y_{s,T}}\le\frac12\|v^1-v^2\|_{Y_{s,T}}$$

for any $v^1,v^2\in B$. Thus, with this choice of r and T, Γ is a contraction in B. Its fixed-point is the unique solution of (2.16) in B.

STEP 3. GLOBAL WELL-POSEDNESS IN $H^0_0(\mathbb{T})$.
Assume that $u_0\in H^0_0(\mathbb{T})$. We first establish (2.18) for $0\le t\le T$. Since $u\in Y_{0,t}$, we have that

$$\begin{aligned}\int_0^t\|uu_x\|^2_{-1}d\tau &\le C\int_0^t\|u^2\|^2d\tau\\ &\le C\int_0^t\|u\|^3\|u_x\|\,d\tau\\ &\le C\sqrt{t}\|u\|^4_{Y_{0,t}}.\end{aligned}$$

Thus each term in (2.16) belongs to $L^2(0,t,H^{-1}(\mathbb{T}))$. Scaling in (2.16) by u yields

$$\int_0^t\langle u_t+(\mathcal{H}-\varepsilon)u_{xx}+uu_x+G(D(Gu)),u\rangle_{H^{-1}(\mathbb{T}),H^1(\mathbb{T})}d\tau=0.$$

We have that for a.e. $\tau\in(0,t)$

$$\begin{aligned}&\langle(\mathcal{H}-\varepsilon)u_{xx},u\rangle_{H^{-1}(\mathbb{T}),H^1(\mathbb{T})}=-((\mathcal{H}-\varepsilon)u_x,u_x)=\varepsilon\|u_x\|^2,\\ &\langle uu_x,u\rangle_{H^{-1}(\mathbb{T}),H^1(\mathbb{T})}=(uu_x,u)=0,\\ &\langle G(D(Gu)),u\rangle_{H^{-1}(\mathbb{T}),H^1(\mathbb{T})}=(G(D(Gu)),u)=\|D^{\frac12}(Gu)\|^2.\end{aligned}$$

(2.18) follows at once, and we infer that $\|u(t)\|\le\|u_0\|$. Using the standard extension argument, one sees that u is defined on $\mathbb{R}^+$ with $u\in Y_{0,T}$ for all $T>0$. Furthermore, with the constants C_0 and C_1 given in Step 2 for $s=0$ and $T=(8C_0C_1\|u_0\|)^{-4}$, we obtain

$$\|u(nT+\cdot)\|_{Y_{0,T}}\le 2C_0\|u(nT)\|\le 2C_0\|u_0\|.$$

Step 4. Global Well-Posedness in $H_0^2(\mathbb{T})$.
Pick any $u_0 \in H_0^2(\mathbb{T})$. By Proposition 2.4 and Step 3, (2.16) admits a unique solution $u \in Y_{0,T}$ for each $T > 0$, which belongs to Y_{2,T_0} for some $T_0 > 0$. We just need to show that T_0 may be taken as large as desired. Let $v = u_t$. If $u \in Y_{2,T}$, then $v \in Y_{0,T}$ and it satisfies

$$v_t + (\mathcal{H} - \varepsilon)v_{xx} + (uv)_x = -G(D(Gv)), \qquad v(0) = v_0 \tag{2.28}$$

where

$$v_0 := -\left\{(\mathcal{H} - \varepsilon)u_{0,xx} + u_0 u_{0,x} + G(D(Gu_0))\right\} \in H_0^0(\mathbb{T}).$$

We may write (2.28) in its integral form

$$v(t) = \mathcal{S}(t)v_0 - \int_0^t \mathcal{S}(t-s)(uv)_x(s)ds.$$

Let $\Gamma(w)(t) = \mathcal{S}(t)v_0 - \int_0^t \mathcal{S}(t-s)(uw)_x(s)ds$ for $w \in Y_{0,T}$. Computations similar to those in Step 2 lead to

$$\begin{aligned} \|\Gamma w\|_{Y_{0,T}} &\leq C_0\|v_0\| + C_1 T^{\frac14}\|u\|_{Y_{0,T}}\|w\|_{Y_{0,T}}, \\ \|\Gamma(w^1) - \Gamma(w^2)\|_{Y_{0,T}} &\leq C_1 T^{\frac14}\|u\|_{Y_{0,T}}\|w^1 - w^2\|_{Y_{0,T}} \end{aligned}$$

where the constants C_0 and C_1 depend only on ε for $T < 1$. Therefore Γ contracts in $B = \{w \in Y_{0,\theta};\ \|w\|_{Y_{0,\theta}} \leq r := 2C_0\|v_0\|\}$, provided that

$$C_1\theta^{\frac14}\|u\|_{Y_{0,\theta}} \leq \frac{1}{2}.$$

Its fixed point gives the unique solution of the integral equation in B. Pick θ fulfilling

$$\theta < \min\{(8C_0C_1\|u_0\|)^{-4}, 1\}.$$

Then, from Step 2, we have that

$$\|u(n\theta + \cdot)\|_{Y_{0,\theta}} \leq 2C_0\|u_0\|$$

for all $n \in \mathbb{N}$ and that w may be extended to $[n\theta, (n+1)\theta]$ inductively by using the contraction mapping theorem (replacing v_0 by $w(\theta)$, $w(2\theta)$, etc.). Therefore, w is defined on $\mathbb{R}^+$ and it holds

$$\|w(n\theta + \cdot)\|_{Y_{0,\theta}} \leq 2C_0\|w(n\theta)\| \leq (2C_0)^{n+1}\|v_0\|. \tag{2.29}$$

By uniqueness of the solution of the integral equation, we have that $v(t) = w(t)$ as long as $0 < t < T$ and $v \in Y_{0,T}$. (2.29) shows that $\|v(t)\| = \|w(t)\|$ is uniformly bounded on compact sets of $\mathbb{R}^+$, namely

$$\|v\|_{Y_{0,T}} \leq C(T, \|u_0\|)\|v_0\|.$$

The same is true for $\|u(t)\|_2$, by (2.16). Indeed, since

$$\|uu_x\| \leq \|u\|_{L^\infty(\mathbb{T})}\|u_x\| \leq \|u\|^{\frac54}\|u_{xx}\|^{\frac34} \leq C_\delta\|u\|^5 + \delta\|u_{xx}\|,$$

we infer from (2.16) that

$$\|(\mathcal{H} - \varepsilon)u_{xx}(t)\| \leq C(T, \|u_0\|)\|u_0\|_2 + C(\|u\| + \|u\|^5) + \delta\|u_{xx}\|$$

hence

$$\|u(t)\|_2 \leq C(T, \|u_0\|)\|u_0\|_2.$$

Using the standard extension argument, one sees that $u(t) \in H_0^2(\mathbb{T})$ for all $t \geq 0$ with $u \in Y_{2,T}$ for all $T > 0$.

Step 5. Smoothing effect from $H_0^0(\mathbb{T})$ to $H_0^2(\mathbb{T})$.
Pick any $u_0 \in H_0^0(\mathbb{T})$. Then the solution u to (2.16) belongs to $Y_{0,1}$. Therefore, for a.e. $t_0 \in (0,1)$, $u(t_0) \in H_0^1(\mathbb{T})$. The solution of (2.16) in $Y_{1,T}$ issued from $u(t_0)$ at $t = 0$ must coincide with $u(t_0 + t)$ in $[0,T]$, by uniqueness of the solution of (2.16) in $Y_{0,T}$. In particular, $u(t_1) \in H_0^2(\mathbb{T})$ for a.e. $t_1 > t_0$. Again by uniqueness we conclude that $u \in C([t_1, +\infty), H_0^2(\mathbb{T}))$ for a.e. $t_1 > 0$, so that

$$u \in C((0,+\infty), H_0^2(\mathbb{T})) \cap C^1((0,+\infty), H_0^0(\mathbb{T})).$$

The proof of Theorem 2.1 is complete. $\square$

The following commutator lemma, used several times in the proof of the property of propagation of regularity, is a periodic version of a result from [10].

Lemma 2.5. *Let $\mathcal{N} \subset \mathbb{Z}$ be a set such that for some constant $C > 0$*

$$\langle n \rangle + \langle k \rangle \leq C \langle n - k \rangle, \qquad \forall n \notin \mathcal{N},\ \forall k \in \mathcal{N}. \tag{2.30}$$

Let P be the projector on the closure of Span$\{e^{ikx};\ k \in \mathcal{N}\}$ in $L^2(\mathbb{T})$, namely

$$P(\sum_{k\in\mathbb{Z}} \hat{u}_k e^{ikx}) = \sum_{k\in\mathcal{N}} \hat{u}_k e^{ikx}.$$

Let $a \in C^\infty(\mathbb{T})$ and let $p \in \mathbb{N},\ q \in \mathbb{N}$. Then there exists some constant $C = C(a,p,q) > 0$ such that for all $v \in L^2(\mathbb{T})$

$$\|\partial_x^p [a, P] \partial_x^q v\| \leq C \|v\|. \tag{2.31}$$

Remark 2.6. *Note that condition* (2.30) *is fulfilled in the following cases: (i) $\mathcal{N} = \mathbb{N}^*$; (ii) $\mathcal{N}$ is a finite set, or the complement of a finite set in $\mathbb{Z}$. It follows that* (2.31) *is true with $P = \mathcal{H} = (-i)(P_{\mathbb{N}^*} - P_{-\mathbb{N}^*})$. Note, however, that condition* (2.30) *and* (2.31) *are not true when $\mathcal{N} = 1 + 2\mathbb{Z}$ (pick e.g. $a(x) = e^{ix}$).*

Proof of Lemma 2.5. Let $\mathcal{N}, a, p$ and q be as in the statement of the lemma, and pick any $v \in C^\infty(\mathbb{T})$. Decompose a and v in using Fourier series

$$v(x) = \sum_{n\in\mathbb{Z}} \hat{v}_n e^{inx}, \qquad a = \sum_{n\in\mathbb{Z}} \hat{a}_n e^{inx},$$

and denote by $1_{\mathcal{N}}$ the characteristic function of $\mathcal{N}$, defined by $1_{\mathcal{N}}(n) = 1$ if $n \in \mathcal{N}$, and 0 otherwise. Then

$$\begin{aligned}
[a,P]v &= a(Pv) - P(av) \\
&= a(\sum_n 1_{\mathcal{N}}(n)\hat{v}_n e^{inx}) - P(\sum_n (\sum_k \hat{a}_{n-k}\hat{v}_k) e^{inx}) \\
&= \sum_n \left(\sum_k \hat{a}_{n-k}\hat{v}_k (1_{\mathcal{N}}(k) - 1_{\mathcal{N}}(n)) \right) e^{inx}.
\end{aligned}$$

Taking derivatives, one obtains

$$\partial_x^p [a,P]\partial_x^q v = \sum_n \left(\sum_k \hat{a}_{n-k}(ik)^q \hat{v}_k (1_{\mathcal{N}}(k) - 1_{\mathcal{N}}(n)) \right) (in)^p e^{inx} =: \Sigma_1 - \Sigma_2$$

where Σ_1 (resp. Σ_2) is the sum over the (n,k) with $n \notin \mathcal{N}$ and $k \in \mathcal{N}$ (resp. with $n \in \mathcal{N}$ and $k \notin \mathcal{N}$). Let us estimate Σ_1 only, the estimate for Σ_2 being similar. Since $a \in C^\infty(\mathbb{T})$, for any $s \in \mathbb{N}$ there exists some constant $C_s > 0$ such that

$$|\hat{a}_l| \le C_s \langle l \rangle^{-s} \qquad \forall l \in \mathbb{Z}. \tag{2.32}$$

Then, for $s > \sup\{2p+1, 2q+1\}$,

$$\begin{aligned}
\|\Sigma_1\|^2_{L^2(\mathbb{T})} &= \| \sum_{n \notin \mathcal{N}} (\sum_{k \in \mathcal{N}} \hat{a}_{n-k} \hat{v}_k (ik)^q (in)^p) e^{inx} \|^2_{L^2(\mathbb{T})} \\
&= C \sum_{n \notin \mathcal{N}} \left| \sum_{k \in \mathcal{N}} \hat{a}_{n-k} \hat{v}_k (ik)^q \right|^2 |n|^{2p} \\
&\le C\|v\|^2 \sum_{n \notin \mathcal{N}} \sum_{k \in \mathcal{N}} \langle n-k \rangle^{-2s} |n|^{2p} |k|^{2q} \\
&\le C\|v\|^2 \sum_{n \notin \mathcal{N}} \sum_{k \in \mathcal{N}} (\langle n \rangle + \langle k \rangle)^{-2s} |n|^{2p} |k|^{2q} \\
&\le C\|v\|^2
\end{aligned}$$

where we used the Cauchy-Schwarz inequality, (2.32) and (2.30). Since $C^\infty(\mathbb{T})$ is dense in $L^2(\mathbb{T})$, the proof is complete. □

The propagation of regularity property we need is as follows.

Proposition 2.7. *Let $a \in C^\infty(\mathbb{T}, \mathbb{R}^+)$, $\varepsilon > 0$, $\alpha \in \mathbb{R}$, $T > 0$, and $R > 0$ be given. Pick any $v_0 \in H^0_0(\mathbb{T})$ with $\|v_0\| \le R$ and let $v \in C([0,T]; H^0_0(\mathbb{T})) \cap L^2(0,T,H^1(\mathbb{T})) \cap C((0,T],H^2(\mathbb{T}))$ be such that*

$$v_t + (\mathcal{H} - \varepsilon)v_{xx} + \alpha v v_x = -G(D(Gv)), \qquad x \in \mathbb{T},\ t \in (0,T) \tag{2.33}$$

$$v(0) = v_0. \tag{2.34}$$

Then there exists some constant $C = C(T) > 0$ (independent of ε, α and R) such that

$$\int_0^T \|D^{\frac{1}{2}} v\|^2 dt \le C(R^2 + \alpha^4 R^6). \tag{2.35}$$

Proof of Proposition 2.7. Pick any $t_0 \in (0,T)$. Let $(f,g)_{L^2_{t,x}} := \int_{t_0}^T \int_{\mathbb{T}} f(x,t)g(x,t)\,dxdt$ denote the scalar product in $L^2(t_0,T,L^2(\mathbb{T}))$. C will denote a constant which may vary from line to line, and which may depend on T, but not on t_0, ε, α and R. Setting $Lv := v_t + \mathcal{H}v_{xx}$, $f := \varepsilon v_{xx} - G(D(Gv))$ and $g := -\alpha v v_x$, we have that

$$Lv = f + g.$$

Pick any $\varphi \in C^\infty(\mathbb{T})$, and set $Av = \varphi(x)v$. Noticing that L is formally skew-adjoint, we have that

$$\begin{aligned}
([L,A]v,v)_{L^2_{t,x}} &= (L(\varphi v) - \varphi(Lv), v)_{L^2_{t,x}} \\
&= (\varphi v, L^* v)_{L^2_{t,x}} + [(\varphi v, v)]_{t_0}^T - (Lv, \varphi v)_{L^2_{t,x}}
\end{aligned}$$

so that

$$|([L,A]v,v)_{L^2_{t,x}}| \le 2|(f+g,\varphi v)_{L^2_{t,x}}| + 2\|\varphi\|_{L^\infty(\mathbb{T})} R^2.$$

We first notice that

$$\begin{aligned}
|(f,\varphi v)_{L^2_{t,x}}| &\le |(v_x, \varepsilon(\varphi v)_x)_{L^2_{t,x}}| + |(D(Gv), G(\varphi v))_{L^2_{t,x}}| \\
&\le C\varepsilon \int_0^T \int_{\mathbb{T}} (|v|^2 + |v_x|^2) dt + C\int_0^T \|D^{\frac12}(Gv)\|^2 dt \\
&\quad + \int_0^T \{|(D(Gv),[G,\varphi]v)| + |(D^{\frac12}(Gv),[D^{\frac12},\varphi](Gv))|\} d\tau \\
&\le CR^2
\end{aligned}$$

where we used (2.18) and classical commutator estimates. (Note that Theorem 2.1 is still true when $\alpha = 1$ is replaced by any value $\alpha \in \mathbb{R}$.) On the other hand

$$|(g,\varphi v)_{L^2_{t,x}}| = |(\alpha v v_x, \varphi v)_{L^2_{t,x}}| = \frac{|\alpha|}{3} |(v^3, \varphi_x)_{L^2_{t,x}}|.$$

From Sobolev embedding and the fact that the L^2-norm is nonincreasing

$$\|v\|_{L^3} \le \|v\|^{\frac12} \|v\|_{L^6}^{\frac12} \le CR^{\frac12} \|v\|_{\frac12}^{\frac12}.$$

Therefore,

$$\begin{aligned}
|(g,\varphi v)_{L^2_{t,x}}| &\le C|\alpha| \int_{t_0}^T \|v\|_{L^3}^3 dt \\
&\le C|\alpha| R^{\frac32} T^{\frac14} \left(\int_{t_0}^T \|v\|_{\frac12}^2 dt \right)^{\frac34} \\
&\le C\delta^{-3}\alpha^4 R^6 T + \delta \int_{t_0}^T \|D^{\frac12} v\|^2 dt
\end{aligned}$$

where $\delta > 0$ will be chosen later on. On the other hand

$$\begin{aligned}
[L,A]v &= [\mathcal{H}\partial_x^2, \varphi]v \\
&= \mathcal{H}\big((\partial_x^2\varphi)v + 2(\partial_x\varphi)(\partial_x v) + \varphi\partial_x^2 v\big) - \varphi\mathcal{H}\partial_x^2 v \\
&= [\mathcal{H},\varphi]\partial_x^2 v + \mathcal{H}\big((\partial_x^2\varphi)v\big) + 2[\mathcal{H},\partial_x\varphi]\partial_x v + 2(\partial_x\varphi)\mathcal{H}\partial_x v. \qquad (2.36)
\end{aligned}$$

It follows from Lemma 2.5 and Remark 2.6 that

$$\begin{aligned}
&|([\mathcal{H},\varphi]\partial_x^2 v, v)_{L^2_{t,x}}| + |\big(\mathcal{H}((\partial_x^2\varphi)v), v\big)_{L^2_{t,x}}| + |\big([\mathcal{H},\partial_x\varphi]\partial_x v, v\big)_{L^2_{t,x}}| \\
&\quad \le C\|v\|^2_{L^2(0,T;L^2(\mathbb{T}))} \\
&\quad \le CR^2. \qquad (2.37)
\end{aligned}$$

Therefore

$$|(\partial_x\varphi\mathcal{H}\partial_x v, v)_{L^2_{t,x}}| \le C(R^2 + \delta^{-3}\alpha^4 R^6) + \delta\int_{t_0}^T \|D^{\frac12}v\|^2 dt.$$

Let $b \in C_0^\infty(\omega)$, where $\omega = \{x \in \mathbb{T};\ a(x) > 0\}$. Then $b = a\tilde b$ with $\tilde b \in C_0^\infty(\omega)$ and

$$\begin{aligned}
\int_{t_0}^T \|D^{\frac12}(bv)\|^2 dt &\le 2\int_{t_0}^T \big(\|[D^{\frac12},\tilde b](av)\|^2 + \|\tilde b D^{\frac12}(av)\|^2\big)dt \\
&\le C\int_{t_0}^T (\|v\|^2 + \|D^{\frac12}(av)\|^2)dt \\
&\le C\int_0^T \big(\|v\|^2 + \|D^{\frac12}(Gv)\|^2 + \|D^{\frac12}a\|^2 |\int_{\mathbb{T}} a(y)v(y,t)\,dy|^2\big)dt \\
&\le CR^2. \qquad (2.38)
\end{aligned}$$

Pick any $x_0 \in \mathbb{T}$. Then $b^2(x) - b^2(x-x_0) = \partial_x\varphi$ for some $\varphi \in C^\infty(\mathbb{T})$. Noticing that $\mathcal{H}\partial_x = D$, we have that

$$\begin{aligned}
|(b^2(x)\mathcal{H}\partial_x v, v)_{L^2_{t,x}}| &= |(bDv, bv)_{L^2_{t,x}}| \\
&\le |([b,D]v, bv)_{L^2_{t,x}}| + |(D(bv), bv)_{L^2_{t,x}}| \\
&\le C\|v\|^2_{L^2(0,T;L^2(\mathbb{T}))} + \int_{t_0}^T \|D^{\frac12}(bv)\|^2 dt \\
&\le CR^2
\end{aligned}$$

by (2.38). It follows that

$$|(b^2(x-x_0)Dv, v)_{L^2_{t,x}}| \le C(R^2 + \delta^{-3}\alpha^4 R^6) + \delta\int_{t_0}^T \|D^{\frac12}v\|^2 dt.$$

Using a partition of unity and choosing $\delta > 0$ small enough, we infer that

$$|(Dv, v)_{L^2_{t,x}}| \le C(R^2 + \alpha^4 R^6) + \frac12\int_{t_0}^T \|D^{\frac12}v\|^2 dt.$$

This gives

$$\int_{t_0}^T \|D^{\frac12}v\|^2 dt \le C(R^2 + \alpha^4 R^6),$$

where $C = C(T)$. Letting $t_0 \to 0$ yields the result. □

A unique continuation property is also required.

Proposition 2.8. *Let $\alpha \in \mathbb{R}$, $\varepsilon \geq 0$, $c \in L^2(0,T)$, and $u \in L^2(0,T;H_0^0(\mathbb{T}))$ be such that*

$$u_t + (\mathcal{H} - \varepsilon)u_{xx} + \alpha u u_x = 0 \qquad \text{in } \mathbb{T} \times (0,T), \tag{2.39}$$

$$u(x,t) = c(t) \qquad \text{for a.e. } (x,t) \in (a,b) \times (0,T) \tag{2.40}$$

for some numbers $T > 0$ and $0 \leq a < b \leq 2\pi$. Then $u(x,t) = 0$ for a.e. $(x,t) \in \mathbb{T} \times (0,T)$.

Proof. From (2.40), we obtain that $u_{xx}(x,t) = (uu_x)(x,t) = 0$ for a.e. $(x,t) \in (a,b) \times (0,T)$. Thus, by using (2.39),

$$\mathcal{H}u_{xx} = -u_t = -c_t \qquad \text{in } (a,b) \times (0,T).$$

Therefore, for almost every $t \in (0,T)$, it holds

$$u_{xxx}(\cdot,t) \in H^{-3}(\mathbb{T}), \tag{2.41}$$

$$u_{xxx}(\cdot,t) = 0 \quad \text{in } \ (a,b), \tag{2.42}$$

$$\mathcal{H}u_{xxx}(\cdot,t) = 0 \quad \text{in } \ (a,b). \tag{2.43}$$

Pick a time t as above, and set $v = u_{xxx}(\cdot,t)$. Decompose v as

$$v(x) = \sum_{k \in \mathbb{Z}} \hat{v}_k e^{ikx},$$

the convergence of the Fourier series being in $H^{-3}(\mathbb{T})$. Then in (a,b)

$$0 = iv - \mathcal{H}v = 2i \sum_{k>0} \hat{v}_k e^{ikx}.$$

Since v is real-valued, we also have that $\hat{v}_{-k} = \overline{\hat{v}_k}$ for all k. The following lemma for Fourier series is needed.

Lemma 2.9. *Let $s \in \mathbb{R}$ and let $v(x) = \sum_{k \geq 0} \hat{v}_k e^{ikx}$ be such that $v \in H^s(\mathbb{T})$ and $v = 0$ in (a,b). Then $v \equiv 0$.*

Proof of Lemma 2.9. It is clearly sufficient to prove the property for $s = -p$, where $p \in \mathbb{N}$. Let us proceed by induction on p. Assume first that $p = 0$. Then

$$\sum_{k \geq 0} |\hat{v}_k|^2 < \infty. \tag{2.44}$$

Introduce the set $U = \{z \in \mathbb{C};\ |z| < 1\}$ and the Hardy space (see e.g. [43])

$$\mathbf{H}^2(U) = \{f : U \to \mathbb{C};\ f \text{ is holomorphic in } U \text{ and } \limsup_{r \to 1^-} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 d\theta < \infty\}$$

Let $f(z) = \sum_{k \geq 0} \hat{v}_k z^k$. Then, by [43, Thm 17.10] and (2.44), we have that $f \in \mathbf{H}^2(U)$. On the other hand, by [43, Thm 17.10 and Thm 17.18], it holds that

$$f^*(e^{i\theta}) := \lim_{r \to 1^-} f(re^{i\theta}) \text{ exists for a.e. } \theta \in (0,2\pi); \tag{2.45}$$

$$f^*(e^{i\theta}) = \sum_{k \geq 0} \hat{v}_k e^{ik\theta} = v(\theta) \qquad \text{in } L^2(\mathbb{T}); \tag{2.46}$$

$$\text{If } f \not\equiv 0, \quad \text{then } f^*(e^{i\theta}) \neq 0 \ \text{ for a.e. } \theta \in (0,2\pi). \tag{2.47}$$

Since

$$f^*(e^{i\theta}) = v(\theta) = 0 \qquad \text{for a.e. } \theta \in (a,b),$$

it follows from (2.47) that $f \equiv 0$. Therefore $\hat{v}_k = 0$ for all $k \geq 0$, hence $v \equiv 0$. This gives the result for $p = 0$. Assume now that the result has been proved for $s = -p$ for some $p \in \mathbb{N}$, and pick any $v \in H^{-p-1}(\mathbb{T})$, decomposed as $v(x) = \sum_{k\geq 0} \hat{v}_k e^{ikx}$, and such that $v \equiv 0$ in (a,b). Let $w(x) = \sum_{k>0} \frac{\hat{v}_{k-1}}{ik} e^{ikx}$. Then $w \in H^{-p}(\mathbb{T})$ and

$$w_x = \sum_{k>0} \hat{v}_{k-1} e^{ikx} = e^{ix} v,$$

so $w_x = 0$ on (a,b) and we have, for some constant $C \in \mathbb{C}$,

$$w(x) = C \quad \text{on} \quad (a,b). \tag{2.48}$$

Introducing the function $\tilde{w}(x) = w(x) - C$, we infer from (2.48) and the induction hypothesis that $\tilde{w} \equiv 0$ on $\mathbb{T}$, which yields $v \equiv 0$ on $\mathbb{T}$. This completes the proof of Lemma 2.9. □

With Lemma 2.9 we infer that for a.e. $t \in (0,T)$, $u_{xxx}(.,t) = 0$ in $\mathbb{T}$, hence with (2.40) $u(x,t) = c(t)$ a.e. in $\mathbb{T} \times (0,T)$. From (2.39) we infer that $c_t = 0$, which, combined with the fact that $u \in L^2(0,T;H_0^0(\mathbb{T}))$, gives that $u(x,t) = 0$ a.e. in $\mathbb{T} \times (0,T)$. The proof of Proposition 2.8 is complete. □

We are now in a position to state a stabilization result for the ε-BO equation. We stress that the decay rate does not depend on ε.

Theorem 2.10. *Let $R > 0$. There exist some numbers $\lambda > 0$ and $C > 0$ such that for any $\varepsilon \in (0,1]$ and any $u_0 \in H_0^0(\mathbb{T})$ with $\|u_0\| \leq R$, the solution u of (2.16) satisfies*

$$\|u(t)\| \leq Ce^{-\lambda t}\|u_0\| \qquad \forall t \geq 0.$$

Proof. Note that $\|u(t)\|$ is nonincreasing by (2.18), so that the exponential decay is ensured if $\|u((n+1)T)\| \leq \kappa \|u(nT)\|$ for some $\kappa < 1$. To prove the theorem, it is thus sufficient (with (2.18)) to establish the following observability inequality: for any $T > 0$ and any $R > 0$ there exists some constant $C(T,R) > 0$ such that for any $\varepsilon \in (0,1]$ and any $u_0 \in H_0^0(\mathbb{T})$ with $\|u_0\| \leq R$, it holds

$$\|u_0\|^2 \leq C\left(\varepsilon \int_0^T \|u_x(t)\|^2 dt + \int_0^T \|D^{\frac{1}{2}}(Gu)\|^2 dt\right), \tag{2.49}$$

where u denotes the solution of (2.16). Fix any $T > 0$ and any $R > 0$, and assume that (2.49) fails. Then there exist a sequence (u_0^n) in $H_0^0(\mathbb{T})$ and a sequence (ε^n) in $(0,1]$ such that for each n we have $\|u_0^n\| \leq R$, and

$$\|u_0^n\|^2 > n\left(\varepsilon^n \int_0^T \|u_x^n(t)\|^2\, dt + \int_0^T \|D^{\frac{1}{2}}(G\,u^n)\|^2 dt\right).$$

Let $\alpha^n = \|u_0^n\| \in (0,R]$. Extracting a sequence if needed, we may assume that $\alpha^n \to \alpha \in [0,R]$ and $\varepsilon^n \to \varepsilon \in [0,1]$. Let $v^n = u^n/\alpha^n$. Then v^n solves

$$v_t^n + (\mathcal{H} - \varepsilon^n)v_{xx}^n + \alpha^n v^n v_x^n = -G(D(Gv^n)), \qquad v^n(0) = v_0^n \tag{2.50}$$

with $v_0^n \in H_0^0(\mathbb{T})$ and $\|v_0^n\| = 1$. Again, we have that

$$\frac{1}{2}\|v^n(t)\|^2 + \varepsilon^n \int_0^t \|v_x^n\|^2 d\tau + \int_0^t \|D^{\frac{1}{2}}(Gv^n)\|^2 d\tau = \frac{1}{2}\|v_0^n\|^2 \qquad \forall t > 0, \tag{2.51}$$

$$1 = \|v_0^n\|^2 > n\left(\varepsilon^n \int_0^T \|v_x^n(t)\|^2 dt + \int_0^T \|D^{\frac{1}{2}}(Gv^n)\|^2\, dt\right). \tag{2.52}$$

We infer from Proposition 2.7 that

$$\int_0^T \|D^{\frac{1}{2}} v^n\|^2 dt \le C. \tag{2.53}$$

This yields

$$\|G(D(Gv^n))\|_{L^2(0,T;H^{-\frac{1}{2}}(\mathbb{T}))} + \|(\mathcal{H} - \varepsilon)v_{xx}^n\|_{L^2(0,T;H^{-\frac{3}{2}}(\mathbb{T}))} \le C.$$

On the other hand, for any $\delta > 0$

$$\|v^n v_x^n\|_{H^{-\frac{3}{2}-\delta}(\mathbb{T})} \le C\|(v^n)^2\|_{H^{-\frac{1}{2}-\delta}(\mathbb{T})} \le C\|(v^n)^2\|_{L^1(\mathbb{T})} \le C\|v^n\|^2 \le C$$

thus

$$\|\alpha^n v^n v_x^n\|_{L^2(0,T;H^{-\frac{3}{2}-\delta}(\mathbb{T}))} \le C.$$

It follows that (v_t^n) is bounded in $L^2(0,T;H^{-\frac{3}{2}-\delta}(\mathbb{T}))$. Combined with (2.53) and Aubin-Lions' lemma, this gives that for a subsequence still denoted by (v^n), we have

$$\begin{aligned} v^n &\to v \qquad \text{in } L^2(0,T;H^\alpha(\mathbb{T})) \qquad \forall \alpha < \frac{1}{2}, \\ v^n &\to v \qquad \text{in } L^2(0,T;H^{\frac{1}{2}}(\mathbb{T})) \text{ weak}, \\ v^n &\to v \qquad \text{in } L^\infty(0,T;L^2(\mathbb{T})) \text{ weak}* \end{aligned}$$

for some function $v \in L^2(0,T;H_0^{\frac{1}{2}}(\mathbb{T})) \cap L^\infty(0,T;L^2(\mathbb{T}))$. In particular,

$$(v^n)^2 \to v^2 \qquad \text{in } L^1(\mathbb{T} \times (0,T)).$$

Letting $n \to \infty$ in (2.52), we obtain that

$$\int_0^T \|D^{\frac{1}{2}}(Gv)\|^2 dt = 0,$$

hence $Gv = 0$ a.e. on $\mathbb{T} \times (0,T)$. Recall that $\omega = \{x \in \mathbb{T};\ a(x) > 0\}$. Then

$$v(x,t) = \int_{\mathbb{T}} a(y)v(y,t)\, dy =: c(t) \qquad \text{for a.e. } (x,t) \in \omega \times (0,T).$$

Note that $c \in L^\infty(0,T)$. Taking the limit in (2.50) gives

$$\begin{cases} v_t + (\mathcal{H} - \varepsilon)v_{xx} + \alpha v v_x = 0, & \text{in } \mathbb{T} \times (0,T), \\ v(x,t) = c(t) & \text{for a.e. } (x,t) \in \omega \times (0,T). \end{cases}$$

It follows from Proposition 2.8 that $v \equiv 0$. Thus, extracting a subsequence still denoted by (v^n), we have that $v^n(\cdot,t) \to 0$ in $L^2(\mathbb{T})$ for a.e. $t \in (0,T)$. Using (2.51)-(2.52), we infer that $v_0^n \to 0$ in $L^2(\mathbb{T})$. This contradicts the fact that $\|v_0^n\| = 1$ for all n. □

We are now in a position to define the weak solutions of (2.11) obtained by the method of vanishing viscosity, and to state the corresponding exponential stability property.

Definition 2.11. *For $u_0 \in H^0_0(\mathbb{T})$, we call a* weak solution of (2.11) in the sense of vanishing viscosity *any function $u \in C_w(\mathbb{R}^+, H^0_0(\mathbb{T}))$ with $u \in L^2(0,T,H^{\frac12}(\mathbb{T}))$ for all $T>0$ which solves* (2.11) *(in the distributional sense) and such that for some sequence $\varepsilon^n \searrow 0$ we have for all $T>0$*

$$\begin{aligned} u^n &\to u \qquad \text{in } L^\infty(0,T,H^0_0(\mathbb{T})) \text{ weak}*, \\ u^n &\to u \qquad \text{in } L^2(0,T,H^{\frac12}_0(\mathbb{T})) \text{ weak} \end{aligned}$$

where u^n solves (2.16) *for $\varepsilon=\varepsilon^n$.*

The main result in this section is the following

Theorem 2.12. *For any $u_0 \in H^0_0(\mathbb{T})$ there exists (at least) one weak solution of* (2.11) *in the sense of vanishing viscosity. On the other hand, for all $R>0$ there exist some positive constants $\lambda=\lambda(R)$ and $C=C(R)$ such that for any weak solution $u(t)$ of* (2.11) *in the sense of vanishing viscosity, it holds*

$$\|u(t)\| \le Ce^{-\lambda t}\|u_0\| \qquad \forall t \ge 0 \tag{2.54}$$

whenever $\|u_0\| \le R$.

Proof. Pick $R>0$ and $u_0 \in H^0_0(\mathbb{T})$ with $\|u_0\| \le R$. Pick any sequence $\varepsilon^n \searrow 0$ and let $u^n(t)$ denote the solution of

$$u^n_t + (\mathcal{H}-\varepsilon^n)u^n_{xx} + u^n u^n_x = -G(DGu^n), \quad u^n(0)=u_0. \tag{2.55}$$

It follows from (2.18) and (2.35) that

$$\begin{aligned} &\|u^n\|_{L^\infty(0,T,H^0_0(\mathbb{T}))} \le R, \\ &\|u^n\|_{L^2(0,T,H^{\frac12}_0(\mathbb{T}))} \le C(T,R). \end{aligned}$$

Using a diagonal process, we obtain that for a subsequence, still denoted by (u^n), we have for all $T>0$

$$u^n \to u \qquad \text{in } L^\infty(0,T,H^0_0(\mathbb{T})) \ \text{ weak}*, \tag{2.56}$$

$$u^n \to u \qquad \text{in } L^2(0,T,H^{\frac12}_0(\mathbb{T})) \text{ weak} \tag{2.57}$$

for some function $u \in L^\infty(\mathbb{R}^+, H^0_0(\mathbb{T})) \cap L^2_{loc}(\mathbb{R}^+, H^{\frac12}_0(\mathbb{T}))$. The same argument as in the proof of Theorem 2.10 shows that $\{u^n_t\}$ is bounded in $L^2(0,T;H^{-\frac32-\delta}(\mathbb{T}))$ for all $\delta>0$. Combined with (2.56)-(2.57) and Aubin-Lions' lemma, this shows that

$$u^n \to u \ \text{ in } \ L^2(\mathbb{T}\times(0,T)) \text{ and in } C([0,T],H^{-\delta}_0(\mathbb{T}))$$

for all $T>0$ and all $\delta>0$. On the other hand, $u \in C([0,T],H^{-\delta}(\mathbb{T}))$ for all $T>0$ and all $\delta>0$, which, combined to (2.56), yields $u \in C_w(\mathbb{R}^+, H^0_0(\mathbb{T}))$ (the space of weakly continuous functions from $\mathbb{R}^+$ to $H^0_0(\mathbb{T})$). By letting $n\to\infty$ in (2.55), we see that u solves (2.11). Thus u is a weak solution of (2.11) in the sense of vanishing viscosity. On the other hand, from Theorem 2.10 we have that

$$\|u^n(t)\| \le Ce^{-\lambda t}\|u_0\|, \quad \forall t\ge 0, \ \forall n \ge 0.$$

where $C = C(R)$, $\lambda = \lambda(R)$. Letting $n \to \infty$ in the above estimate yields (2.54). Note also that $\|u(t)\| \le \|u_0\|$ for all $t \ge 0$, since the same estimate holds for the u^n's and $u \in C_w(\mathbb{R}^+, H_0^0(\mathbb{T}))$. $\Box$

2.2. **Local stabilization in $H_0^s(\mathbb{T})$.**

2.2.1. *Main results.* Let again a and G be as in (2.10) and (2.12), respectively. For $s \ge 0$ and $T > 0$, let

$$Z_{s,T} = C([0,T], H_0^s(\mathbb{T})) \cap L^2(0,T,H_0^{s+\frac{1}{2}}(\mathbb{T})) \tag{2.58}$$

be endowed with the norm

$$\|v\|_{Z_{s,T}} = \|v\|_{L^\infty(0,T,H^s(\mathbb{T}))} + \|v\|_{L^2(0,T,H^{s+\frac{1}{2}}(\mathbb{T}))}.$$

We are concerned here with the stability properties of the BO equation with localized damping (2.11) in the space $H_0^s(\mathbb{T})$ for $s > 0$. Our first aim is to prove the local well-posedness of (2.11) in $H_0^s(\mathbb{T})$ for $s > 1/2$.

Theorem 2.13. *Let $s \in (\frac{1}{2}, 2]$. Then there exists $\rho > 0$ such that for any $u_0 \in H_0^s(\mathbb{T})$ with $\|u_0\|_s < \rho$, there exists some time $T > 0$ such that* (2.11) *admits a unique solution in the space $Z_{s,T}$.*

The proof of Theorem 2.13 rests on the smoothing effect due to the damping term, namely

$$\int_0^T \|e^{-t(\mathcal{H}\partial_x^2 + GDG)} u_0\|_{\frac{1}{2}}^2 \, dt \le C\|u_0\|^2. \tag{2.59}$$

In [37], the semi-global exponential stability of the Korteweg-de Vries on a bounded domain $(0,L)$ with a localized damping was first established in $L^2(0,L)$, and next extended to $\{u \in H^3(0,L);\quad u(0) = u(L) = u_x(L) = 0\}$ by using the Kato smoothing effect in the equation fulfilled by the time derivative of the solution. As the smoothing effect (2.59) is much weaker, that argument cannot be used. The semi-global exponential stability of (2.11) in $H_0^0(\mathbb{T})$, if true, is thus open. However, a local exponential stability in $H_0^s(\mathbb{T})$ for $s > 1/2$ can be derived.

Theorem 2.14. *Let $s \in (\frac{1}{2}, 2]$. Then there exist some numbers $\rho > 0$, $\lambda > 0$ and $C > 0$ such that for any $u_0 \in H_0^s(\mathbb{T})$ with $\|u_0\|_s < \rho$, there is a (unique) solution $u : \mathbb{R}^+ \to H_0^s(\mathbb{T})$ of* (2.11) *with $u \in Z_{s,T}$ for all $T > 0$ and such that*

$$\|u(t)\|_s \le Ce^{-\lambda t}\|u_0\|_s \qquad \forall t \ge 0. \tag{2.60}$$

The proofs of Theorem 2.13 and Theorem 2.14 are given in the next sections.

2.2.2. *Linear Theory.* In this section, we focus on the well-posedness and the smoothing property of the linearized BO equation with localized damping:

$$u_t + \mathcal{H}u_{xx} + GDGu = 0, \qquad u(0) = u_0. \tag{2.61}$$

Let $s \in \mathbb{R}$ and let $Au = -(\mathcal{H}u_{xx} + GDGu)$ with domain $\mathcal{D}(A) = H_0^{s+2}(\mathbb{T}) \subset H_0^s(\mathbb{T})$. Our first result is the

Lemma 2.15. *A generates a continuous semigroup in $H_0^s(\mathbb{T})$, denoted by $(S(t))_{t\geq 0}$.*

Proof. Let $C = C(s)$ be the constant in Claim 1. Clearly, $A - C$ is a densely defined closed operator in $H_0^s(\mathbb{T})$. Furthermore, by Claim 1,

$$(Au - Cu, u)_s \leq -\|D^{\frac{1}{2}}(Gu)\|_s^2 \quad \forall u \in H_0^{s+2}(\mathbb{T}),$$

which shows that $A-C$ is dissipative. It is easily verified that $D(A^*) = D(A) = H_0^{s+2}(\mathbb{T})$. Thus

$$(A^*u - Cu, u)_s = (u, Au - Cu)_s \leq 0 \quad \forall u \in H_0^{s+2}(\mathbb{T}),$$

so that $A^* - C$ is dissipative too. Thus, $A - C$ generates a semigroup of contractions in $H_0^s(\mathbb{T})$ by [35, Cor. 4.4, p. 15]. $\square$

Now we turn our attention to the smoothing effect.

Proposition 2.16. *Let $s \geq 0$, $v_0 \in H_0^s(\mathbb{T})$ and $g \in L^2(0, T, H_0^{s-\frac{1}{2}}(\mathbb{T}))$. Then the solution v of*

$$v_t + \mathcal{H}v_{xx} + GDGv = g, \qquad v(0) = v_0 \tag{2.62}$$

satisfies $v \in Z_{s,T}$ with

$$\|v\|_{Z_{s,T}} \leq C(s,T)\left(||v_0||_s + ||g||_{L^2(0,T,H^{s-\frac{1}{2}}(\mathbb{T}))}\right), \tag{2.63}$$

$C(s,T)$ being nondecreasing in T.

Proof. Let us assume first that $s = 0$. To have enough regularity in the computations, we assume that $v_0 \in H_0^2(\mathbb{T})$ and that $g \in C([0,T], H_0^2(\mathbb{T}))$, so that the solution v of (2.62) satisfies $v \in C([0,T], H_0^2(\mathbb{T})) \cap C^1([0,T], H_0^0(\mathbb{T}))$. We now proceed as in the proof of Proposition 2.7. We set $Lv = v_t + \mathcal{H}v_{xx}$, $f = -GDGv$, so that $Lv = f + g$. Pick any $\varphi \in C^\infty(\mathbb{T})$, and let $Av = \varphi(x)v$. Then

$$|\int_0^T ([L,A]v, v)\, dt| \leq 2|\int_0^T (f + g, \varphi v) dt| + \|\varphi\|_{L^\infty}\left(\|v_0\|^2 + \|v(T)\|^2\right)\cdot$$

Scaling in (2.62) by v yields

$$\begin{aligned}
\frac{1}{2}\|v(t)\|^2 + \int_0^t \|D^{\frac{1}{2}}Gv\|^2 d\tau &= \frac{1}{2}\|v_0\|^2 + \int_0^t (g, v) d\tau \\
&\leq \frac{1}{2}\|v_0\|^2 + \int_0^T \|g\|_{-\frac{1}{2}}\|v\|_{\frac{1}{2}} dt.
\end{aligned}$$

This yields

$$\|v\|^2_{L^\infty(0,T,H^0(\mathbb{T}))} + \int_0^T \|D^{\frac{1}{2}}(Gv)\|^2 d\tau \leq \frac{3}{2}\|v_0\|^2 + 3\int_0^T \|g\|_{-\frac{1}{2}}\|v\|_{\frac{1}{2}} dt. \tag{2.64}$$

Computations similar to those in Proposition 2.7 give that

$$\begin{aligned}
|\int_0^T (f + g, \varphi(x)v)\, d\tau| &\leq C\|\varphi\|_1 \int_0^T (\|D^{\frac{1}{2}}(Gv)\|^2 + \|v\|^2 + \|g\|_{-\frac{1}{2}}\|v\|_{\frac{1}{2}}) dt \\
&\leq C(T, \|\varphi\|_1)\left(\|v_0\|^2 + \int_0^T \|g\|_{-\frac{1}{2}}\|v\|_{\frac{1}{2}} dt\right),
\end{aligned}$$

hence

$$|\int_0^T ([L,A]v,v)dt| \le C(T,\|\varphi\|_1)\left(\|v_0\|^2+\int_0^T \|g\|_{-\frac12}\|v\|_{\frac12}dt\right).$$

Combined with (2.36)-(2.37), the last inequality gives

$$|\int_0^T (\partial_x\varphi Dv,v)\,dt| \le C(T,\|\varphi\|_1)\left(\|v_0\|^2+\int_0^T \|g\|_{-\frac12}\|v\|_{\frac12}dt\right). \tag{2.65}$$

We pick again $b\in C_0^\infty(\omega)$, where $\omega=\{x\in\mathbb{T};\ a(x)>0\}$ and $x_0\in\mathbb{T}$. Writing again $b^2(x)-b^2(x-x_0)=\partial_x\varphi$, we obtain successively, with (2.38) and (2.64), that

$$\begin{aligned}
\int_0^T \|D^{\frac12}(bv)\|^2dt &\le C(T)\left(\|v_0\|^2+\int_0^T \|g\|_{-\frac12}\|v\|_{\frac12}dt\right),\\
|\int_0^T (b^2Dv,v)dt| &\le C\int_0^T \left(\|v\|^2+\|D^{\frac12}(bv)\|^2\right)dt\\
&\le C(T)\left(\|v_0\|^2+\int_0^T \|g\|_{-\frac12}\|v\|_{\frac12}dt\right)
\end{aligned}$$

and therefore, with (2.65),

$$|\int_0^T (b^2(x-x_0)Dv,v)dt| \le C(T)\left(\|v_0\|^2+\int_0^T \|g\|_{-\frac12}\|v\|_{\frac12}dt\right).$$

Using a partition of unity, this yields

$$\int_0^T \|v\|^2_{\frac12}dt \le C(T)\left(\|v_0\|^2+\int_0^T \|g\|^2_{-\frac12}dt\right)+\frac12\int_0^T \|v\|^2_{\frac12}dt.$$

Combined with (2.64), this gives (2.63) for $s=0$ when $v_0\in H_0^2(\mathbb{T})$ and $g\in C([0,T],H_0^2(\mathbb{T}))$. This is also true for $v_0\in H_0^0(\mathbb{T})$ and $g\in L^1(0,T,H_0^{-\frac12}(\mathbb{T}))$ by density.

Let us now assume that $s\in(0,2]$. Pick again any $v_0\in H_0^2(\mathbb{T})$, $g\in C([0,T],H_0^2(\mathbb{T}))$, and let $v\in C([0,T],H_0^2(\mathbb{T}))\cap C^1([0,T],H_0^0(\mathbb{T}))$ denote the solution of (2.62). Set $w=D^sv$ and $h=D^sg$. Note that

$$D^s(GDGv)=GDGw+Ew$$

with $E=[D^s,G]DGD^{-s}+GD[D^s,G]D^{-s}$. Note that $\|Ew\|\le C\|w\|$ and that w solves

$$w_t+\mathcal{H}w_{xx}+GDGw+Ew=h,\quad w(0)=w_0:=D^sv_0.$$

Since

$$\begin{aligned}
|\int_0^T (\varphi w,Ew)\,dt| &\le C\|\varphi\|_1\|w\|^2_{L^2(0,T,H^0(\mathbb{T}))}\\
&\le C(T,\|\varphi\|_1)\left(\|w_0\|^2+\int_0^T \|h\|_{-\frac12}\|w\|_{\frac12}dt\right),
\end{aligned}$$

we obtain in a similar fashion as above that

$$\|w\|^2_{L^\infty(0,T,H^0(\mathbb{T}))}+\int_0^T \|w\|^2_{\frac12}dt\le C(T)\left(\|w_0\|^2+\int_0^T \|h\|^2_{-\frac12}dt\right),$$

i.e.

$$\|v\|^2_{L^\infty(0,T,H^s(\mathbb{T}))} + \|v\|^2_{L^2(0,T,H^{s+\frac{1}{2}}(\mathbb{T}))} \leq C(T)\left(\|v_0\|_s^2 + \|g\|^2_{L^2(0,T,H^{s-\frac{1}{2}}(\mathbb{T}))}\right). \tag{2.66}$$

Inequality (2.66) and the fact that $v \in C([0,T], H_0^s(\mathbb{T}))$ are also true for $v_0 \in H_0^s(\mathbb{T})$ and $g \in L^2(0,T,H_0^{s-\frac{1}{2}}(\mathbb{T}))$ by density. □

Corollary 2.17. *Let $s \geq 0$ and $B \in \mathcal{L}(H_0^s(\mathbb{T}))$. Then for any $v_0 \in H_0^s(\mathbb{T})$, the solution v of*

$$v_t + \mathcal{H}v_{xx} + GDGv = Bv, \qquad v(0) = v_0 \tag{2.67}$$

fulfills $v \in Z_{s,T}$ with

$$||u||_{Z_{s,T}} \leq C(s,T)||v_0||_s. \tag{2.68}$$

Proof. Since A is the generator of a continuous semigroup on $H_0^s(\mathbb{T})$ and B is a bounded operator on $H_0^s(\mathbb{T})$, $A+B$ is the generator of a continuous semigroup on $H_0^s(\mathbb{T})$ (see e.g. [35, Thm 1.1 p. 76]). Pick any $v_0 \in H_0^s(\mathbb{T})$, and let v denote the solution of (2.67) given by the semigroup generated by $A+B$. Noticing that $g := Bv \in C([0,T]; H_0^s(\mathbb{T}))$, we infer from Proposition 2.16 that $v \in Z_{s,T}$ with

$$\|v\|_{L^\infty(0,T,H^s(\mathbb{T}))} + \|v\|_{L^2(0,T,H^{s+\frac{1}{2}}(\mathbb{T}))} \leq C(s,T)\left(\|v_0\|_s + \sqrt{T}\,\|B\|_{\mathcal{L}(H_0^s(\mathbb{T}))}\|v\|_{L^\infty(0,T;H^s(\mathbb{T}))}\right).$$

Selecting $T_0 > 0$ such that $c(s,T_0)\sqrt{T_0}\|B\|_{\mathcal{L}(H_0^s(\mathbb{T}))} < 1/2$ yields

$$\|v\|_{L^\infty(0,T_0,H^s(\mathbb{T}))} + \|v\|_{L^2(0,T_0,H^{s+\frac{1}{2}}(\mathbb{T}))} \leq 2C(s,T_0)\|v_0\|_s. \tag{2.69}$$

Successive applications of (2.69) on the intervals $[0,T_0]$, $[T_0,2T_0]$,... give (2.68) for any $T>0$. □

2.2.3. *Proof of Theorem 2.13.* Pick any $s \in (\frac{1}{2}, 2]$ and any $T > 0$. Let $u_0 \in H_0^s(\mathbb{T})$. We write (2.11) in its integral form

$$u(t) = S(t)u_0 - \int_0^t S(t-\tau)(uu_x)(\tau)\,d\tau. \tag{2.70}$$

Let $\Gamma(v)(t) = S(t)u_0 - \int_0^t S(t-\tau)(vv_x)(\tau)d\tau$. We have, by Proposition 2.16, that

$$\|\Gamma(v)\|_{Z_{s,T}} \leq C\left(\|u_0\|_s + \|(\frac{v^2}{2})_x\|_{L^2(0,T,H^{s-\frac{1}{2}}(\mathbb{T}))}\right).$$

Clearly, for $u, v \in Z_{s,T}$,

$$\begin{aligned}
\int_0^T \|(uv)_x\|^2_{s-\frac{1}{2}} dt &\leq C\int_0^T \|uv\|^2_{s+\frac{1}{2}} dt \\
&\leq C\int_0^T \big(\|u\|^2_{L^\infty(\mathbb{T})}\|v\|^2_{s+\frac{1}{2}} + \|u\|^2_{s+\frac{1}{2}}\|v\|^2_{L^\infty(\mathbb{T})}\big)dt \\
&\leq C\int_0^T \big(\|u\|^2_s\|v\|^2_{s+\frac{1}{2}} + \|u\|^2_{s+\frac{1}{2}}\|v\|^2_s\big)dt \\
&\leq C\Bigg(\|u\|^2_{L^\infty(0,T,H^s(\mathbb{T}))}\|v\|^2_{L^2(0,T,H^{s+\frac{1}{2}}(\mathbb{T}))} \\
&\qquad + \|v\|^2_{L^\infty(0,T,H^s(\mathbb{T}))}\|u\|^2_{L^2(0,T,H^{s+\frac{1}{2}}(\mathbb{T}))}\Bigg) \\
&\leq C\|u\|^2_{Z_{s,T}}\|v\|^2_{Z_{s,T}},
\end{aligned}$$

where we used the Sobolev embedding $H^s_0(\mathbb{T}) \subset L^\infty(\mathbb{T})$ for $s > 1/2$. Thus, there are some constants $C_0 > 0$ and $C_1 > 0$ such that

$$\begin{aligned}
\|\Gamma(v)\|_{Z_{s,T}} &\leq C_0\|u_0\|_s + C_1\|v\|^2_{Z_{s,T}} \qquad \forall v \in Z_{s,T}, \\
\|\Gamma(v^1) - \Gamma(v^2)\|_{Z_{s,T}} &\leq C_1\big(\|v^1\|_{Z_{s,T}} + \|v^2\|_{Z_{s,T}}\big)\|v^1 - v^2\|_{Z_{s,T}} \qquad \forall v^1, v^2 \in Z_{s,T}.
\end{aligned}$$

Let $B = \{v \in Z_{s,T};\ \|v\|_{Z_{s,T}} \leq R\}$. We choose R in such a way that B is left invariant by Γ and Γ contracts in B, i.e.

$$C_0\|u_0\|_s + C_1R^2 \leq R, \quad \text{and } 2C_1R < 1.$$

It is sufficient to take $R = (4C_1)^{-1}$ and $u_0 \in H^s_0(\mathbb{T})$ with $\|u_0\|_s \leq \rho := R/(2C_0)$. □

2.2.4. *Proof of Theorem 2.14.* We proceed as in [34]. It has been proved that (2.11) is semi-globally exponentially stable in $H^0_0(\mathbb{T})$. Obviously, the same analysis shows that the *linearized* BO equation with localized damping is also exponentially stable in $H^0_0(\mathbb{T})$, i.e.

$$\|S(t)u_0\| \leq Ce^{-\lambda t}\|u_0\| \tag{2.71}$$

for all $u_0 \in H^0_0(\mathbb{T})$ and some constants $C, \lambda > 0$. If $u_0 \in H^2_0(\mathbb{T})$, then $u(t) = S(t)u_0$ solves

$$u_t + \mathcal{H}u_{xx} + GDGu = 0, \quad u(0) = u_0. \tag{2.72}$$

Letting $v = u_t$, v solves also

$$v_t + \mathcal{H}v_{xx} + GDGv = 0, \quad v(0) = v_0 := -(\mathcal{H}u_{0,xx} + GDGu_0). \tag{2.73}$$

(2.71) yields

$$\|v(t)\| = \|S(t)v_0\| \leq Ce^{-\lambda t}\|v_0\|,$$

and thus

$$\|S(t)u_0\|_2 \leq C'e^{-\lambda t}\|u_0\|_2.$$

By interpolation, this shows that for any $s \in [0,2]$, for any $u_0 \in H^s_0(\mathbb{T})$ and for some constant $C > 0$ (independent of s, u_0, and t), it holds

$$\|S(t)u_0\|_s \leq Ce^{-\lambda t}\|u_0\|_s. \tag{2.74}$$

Let $s > 1/2$ and $u_0 \in H_0^s(\mathbb{T})$. For

$$u \in Z_{s,T}([n,n+1]) := C([n,n+1], H_0^s(\mathbb{T})) \cap L^2(n,n+1, H_0^{s+\frac{1}{2}}(\mathbb{T})),$$

let

$$|||u|||_n = \|u\|_{L^\infty(n,n+1,H^s(\mathbb{T}))} + \|u\|_{L^2(n,n+1,H^{s+\frac{1}{2}}(\mathbb{T}))}.$$

Finally, let

$$\|u\|_E = \sup_{n\geq 0} \left(e^{n\lambda} |||u|||_n \right) \leq +\infty.$$

Introduce the space

$$E = \{u \in C(\mathbb{R}^+, H_0^s(\mathbb{T})) \cap L^2_{loc}(\mathbb{R}^+, H_0^{s+\frac{1}{2}}(\mathbb{T}));\ \|u\|_E < \infty\}.$$

Endowed with the norm $\|\cdot\|_E$, E is a Banach space. We search for a solution of (2.11) in a closed ball $B = \{u \in E;\ \|u\|_E \leq R\}$ as a fixed point of the map $\Gamma(v)(t) = S(t)u_0 - \int_0^t S(t-\tau)(vv_x)(\tau)d\tau$. By (2.74), we have

$$\|S(n)u_0\|_s \leq Ce^{-n\lambda}\|u_0\|_s \quad \forall n \geq 0. \tag{2.75}$$

Combined with Proposition 2.16, this gives for some constant $C_0 > 0$

$$|||S(t)u_0|||_n \leq C_0\, e^{-n\lambda}\|u_0\|_s, \tag{2.76}$$

hence

$$\|S(t)u_0\|_E \leq C_0\|u_0\|_s. \tag{2.77}$$

On the other hand, for any $u, v \in E$,

$$|||\int_0^t S(t-\tau)[(uv)_x(\tau)]d\tau|||_n \leq I_1 + I_2$$

with

$$\begin{aligned} I_1 &= |||S(t-n)\int_0^n S(n-\tau)[(uv)_x(\tau)]d\tau|||_n, \\ I_2 &= |||\int_n^t S(t-\tau)[(uv)_x(\tau)]d\tau|||_n \end{aligned}$$

By (2.63) and (2.76),

$$\begin{aligned}
I_1 &\le C\| \int_0^n S(n-\tau)[(uv)_x(\tau)]d\tau\|_s \\
&\le C\sum_{k=1}^n \|S(n-k)\int_{k-1}^k S(k-\tau)[(uv)_x(\tau)]\|_s \\
&\le C\sum_{k=1}^n e^{-(n-k)\lambda}\| \int_{k-1}^k S(k-\tau)[(uv)_x(\tau)]d\tau\|_s \\
&\le C\sum_{k=1}^n e^{-(n-k)\lambda}\|(uv)_x\|_{L^2(k-1,k,H^{s-\frac{1}{2}}(\mathbb{T}))} \\
&\le C\sum_{k=1}^n e^{-(n-k)\lambda}|\!|\!| u|\!|\!|_{k-1}|\!|\!| v|\!|\!|_{k-1} \\
&\le Ce^{-n\lambda}\|u\|_E\|v\|_E.
\end{aligned}$$

On the other hand

$$I_2 \le C\|(uv)_x\|_{L^2(n,n+1,H^{s-\frac{1}{2}}(\mathbb{T}))} \le Ce^{-2n\lambda}\|u\|_E\|v\|_E.$$

We have proved that for some constant $C_1 > 0$

$$|\!|\!| \int_0^t S(t-\tau)[(uv)_x(\tau)]d\tau |\!|\!|_n \le 2C_1 e^{-n\lambda}\|u\|_E\|v\|_E,$$

hence

$$\| \int_0^t S(t-\tau)[(uv)_x(\tau)]d\tau\|_E \le 2C_1\|u\|_E\|v\|_E.$$

Thus

$$\begin{aligned}
\|\Gamma(v)\|_E &\le C_0\|u_0\|_s + C_1\|v\|_E^2, \\
\|\Gamma(v^1)-\Gamma(v^2)\|_E &\le C_1(\|v^1\|_E + \|v^2\|_E)\|v^1-v^2\|_E.
\end{aligned}$$

It follows that Γ contracts in the ball $B = \{u \in E;\ \|u\|_E \le R\}$ if

$$2C_1R < 1, \quad \text{and} \quad C_0\|u_0\|_s + C_1R^2 \le R. \tag{2.78}$$

Let $R = \gamma\rho$ (γ and ρ being determined later), and assume that $\|u_0\|_s \le \rho$. The conditions become

$$2C_1\gamma\rho < 1, \text{ and } C_0 + C_1\gamma^2\rho \le \gamma. \tag{2.79}$$

Pick $\gamma = 2C_0$ and $\rho > 0$ sufficiently small so that (2.79) holds. Then Γ contracts in B. Replacing ρ by $\|u_0\|_s$, we see that the fixed point $u = \Gamma(u)$ satisfies

$$\|u\|_{L^\infty(n,n+1,H^s(\mathbb{T}))} \le e^{-n\lambda}\|u\|_E \le e^{-n\lambda}\gamma\|u_0\|_s.$$

It follows that

$$\|u(t)\|_s \le Ce^{-\lambda t}\|u_0\|_s \quad \forall t \ge 0$$

for some constant $C > 0$, provided that $\|u_0\|_s \le \rho$. □

3. Control of the Benjamin-Ono equation

Let again a and G be as in (2.10) and (2.12), respectively. We now focus on the control properties of the full BO equation. More precisely, we aim to prove the exact controllability of the system

$$u_t + \mathcal{H}u_{xx} + uu_x = Gh, \qquad u(0) = u_0, \tag{3.80}$$

where h is the control input. If the exact controllability of the linearized system is well known (cf. Theorem A), the exact controllability of (3.80) is challenging, as the contraction mapping theorem cannot be applied directly to BO. To overcome that difficulty, we incorporate the feedback $f = -DGu$ into the control input h to obtain a strong enough smoothing effect to apply the contraction principle. Setting

$$h(t) = -DGu(t) + D^{\frac{1}{2}}k(t), \tag{3.81}$$

we are thus led to investigate the controllability of the system

$$u_t + \mathcal{H}u_{xx} + GDGu + uu_x = GD^{\frac{1}{2}}k, \qquad u(0) = u_0. \tag{3.82}$$

We shall derive the following local exact controllability result.

Theorem 3.1. *Let $s \in (\frac{1}{2}, 2]$ and $T > 0$. Then there exists $\delta > 0$ such that for any $u_0, u_1 \in H_0^s(\mathbb{T})$ with*

$$\|u_0\|_s \le \delta, \qquad \|u_1\|_s \le \delta, \tag{3.83}$$

one may find a control $k \in L^2(0,T,H_0^s(\mathbb{T}))$ such that the system (3.82) admits a (unique) solution u in the class $Z_{s,T}$ for which $u(T) = u_1$.

The proof of Theorem 3.1 is done in three steps. In the first step, we prove the exact controllability of the linearized system

$$u_t + \mathcal{H}u_{xx} + GDGu = GD^{\frac{1}{2}}k, \qquad u(0) = u_0, \tag{3.84}$$

in $L_0^2(\mathbb{T})$. In the second step, we prove the exact controllability of (3.84) in $H_0^s(\mathbb{T})$ for all $s > 0$ by following the same approach as in [41]. Finally, in the third part we derive the exact controllability of the full BO equation by using the contraction mapping theorem as e.g. in [36, 38, 41]. Note that Theorem 1.3 follows at once from Theorem 3.1 by letting

$$h = -DGu + D^{\frac{1}{2}}k \in L^2(0,T,H_0^{s-\frac{1}{2}}(\mathbb{T})).$$

Proof of Theorem 3.1.

Step1. Exact controllability of (3.84) in $H_0^0(\mathbb{T})$.
First, the solution of (3.82) belongs to $Z_{s,T}$ for $u_0 \in H_0^s(\mathbb{T})$ and $k \in L^2(0,T,H_0^s(\mathbb{T}))$, according to Proposition 2.16. The adjoint system reads

$$-v_t - \mathcal{H}v_{xx} + GDGv = 0, \qquad v(T) = v_T. \tag{3.85}$$

Scaling in (3.84) by v yields

$$\int_{\mathbb{T}} uv\,dx\Big|_0^T = \int_0^T\int_{\mathbb{T}} kD^{\frac{1}{2}}(Gv)\,dxdt. \tag{3.86}$$

The computations are fully justified when $u_0, v_T \in H_0^2(\mathbb{T})$ and $k \in L^2(0,T,H_0^{\frac{5}{2}}(\mathbb{T}))$, and next extended to the case when $u_0, v_T \in H_0^0(\mathbb{T})$ and $k \in L^2(0,T,H_0^0(\mathbb{T}))$ by density. Following the classical duality approach, we are led to prove the following observability inequality

$$\|v_T\|^2 \leq C \int_0^T \int_{\mathbb{T}} |D^{\frac{1}{2}}(Gv)|^2 dxdt. \tag{3.87}$$

Once (3.87) is proved, the exact controllability of (3.84) follows by noticing that the operator $\Gamma \in \mathcal{L}(H_0^0(\mathbb{T}))$ defined by $\Gamma(v_T) = u(T)$, where u denotes the solution of (3.84) associated with $u_0 = 0$ and $k = D^{\frac{1}{2}}(Gv)$ and v denotes the solution of (3.85), is onto by (3.87) and Lax-Milgram theorem.

Let us prove (3.87) by contradiction. If (3.87) is not true, then one can pick a sequence (v_T^n) in $H_0^0(\mathbb{T})$ such that

$$1 = \|v_T^n\|^2 > n \int_0^T \int_{\mathbb{T}} |D^{\frac{1}{2}}(Gv^n)|^2 dxdt, \tag{3.88}$$

where v^n denotes the solution of (3.85) issued from $v_T = v_T^n$.

Multiplying each term in (3.85) by tv^n and integrating by parts results in

$$\frac{T}{2}\|v_T^n\|^2 = \frac{1}{2}\int_0^T \int_{\mathbb{T}} |v^n|^2 dxdt + \int_0^T \int_{\mathbb{T}} t\,|D^{\frac{1}{2}}(Gv^n)|^2 dxdt. \tag{3.89}$$

Computations similar to those in the proof of Proposition 2.16 (changing t into $\tau := T - t$) give

$$\|v^n\|_{L^2(0,T,H^{\frac{1}{2}}(\mathbb{T}))} \leq C\|v_T^n\|\cdot \tag{3.90}$$

Thus, by (3.85) and (3.90), (v^n) is bounded in $L^2(0,T,H_0^{\frac{1}{2}}(\mathbb{T})) \cap H^1(0,T,H^{-\frac{3}{2}}(\mathbb{T}))$. By Aubin-Lions' lemma, a subsequence of (v^n), still denoted by (v^n), has a strong limit (say v) in $L^2(0,T,H_0^0(\mathbb{T}))$. It follows from (3.88) and (3.89) that (v_T^n) is a Cauchy sequence in $H_0^0(\mathbb{T})$, hence it has a strong limit (say v_T) in $H_0^0(\mathbb{T})$, with $\|v_T\| = 1$. By standard semigroup theory, v^n converges in $C([0,T],H_0^0(\mathbb{T}))$ to the solution of (3.85) associated with v_T, which therefore agrees with v. By (3.88), $D^{\frac{1}{2}}(Gv) \equiv 0$, hence $Gv \equiv 0$. We conclude that v satisfies

$$\begin{aligned} v_t + \mathcal{H}v_{xx} &= 0, \\ Gv &= 0. \end{aligned}$$

It follows from Proposition 2.8 that $v \equiv 0$. In particular $v_T = v(T) = 0$, a property which contradicts the fact that $\|v_T\| = 1$. The proof of (3.87) is achieved.

Step 2. Exact controllability of (3.84) in $H_0^s(\mathbb{T})$.
Picking any number $s > 0$, we aim to prove the exact controllability of (3.84) in $H_0^s(\mathbb{T})$. Notice first that the system (3.85) is (backward) well-posed in $H_0^{-s}(\mathbb{T})$, since the conclusion of Lemma 2.15 is still valid when $\mathcal{H}u_{xx}$ is replaced by $-\mathcal{H}u_{xx}$ in (2.61). Thus, the following estimate holds

$$\|v\|_{L^\infty(0,T,H^{-s}(\mathbb{T}))} \leq C\|v_T\|_{-s}.$$

On the other hand, setting $w = (1-\partial_x^2)^{-\frac{s}{2}}v$, we see that w solves

$$-w_t - \mathcal{H}w_{xx} + GDGw = (1-\partial_x^2)^{-\frac{s}{2}}[(1-\partial_x^2)^{\frac{s}{2}}, GDG]w =: Bw$$
$$w(T) = (1-\partial_x^2)^{-\frac{s}{2}}v_T =: w_T.$$

Note that $B \in \mathcal{L}(H_0^\sigma(\mathbb{T}))$ for all $\sigma \in \mathbb{R}$ (see e.g. [21]). Using computations similar to those to prove Corollary 2.17, we see that

$$\|w\|_{L^2(0,T,H^{\frac{1}{2}}(\mathbb{T}))} \le C\|w_T\|, \tag{3.91}$$

and hence

$$\|v\|_{L^2(0,T,H^{-s+\frac{1}{2}}(\mathbb{T}))} \le C\|v_T\|_{-s}. \tag{3.92}$$

Assuming again that $u_0 = 0$, we first note that (3.86) may be written

$$\langle v_T, u(T)\rangle_{-s,s} = \int_0^T \langle D^{\frac{1}{2}}(Gv), k\rangle_{-s,s} dt,$$

where $\langle\cdot,\cdot\rangle_{-s,s}$ denotes the duality pairing $\langle\cdot,\cdot\rangle_{H_0^{-s}(\mathbb{T}),H_0^s(\mathbb{T})}$. We aim to prove the observability inequality

$$\|v_T\|_{-s}^2 \le C\int_0^T \|Gv\|_{-s+\frac{1}{2}}^2 dt. \tag{3.93}$$

Once (3.93) is proved, the exact controllability of (3.84) in $H_0^s(\mathbb{T})$ follows easily. Indeed, if $\Gamma_{-s} \in \mathcal{L}(H_0^{-s}(\mathbb{T}))$ is defined by $\Gamma_{-s}(v_T) = (1-\partial_x^2)^s u(T)$ where u solves (3.84) with $k = (1-\partial_x^2)^{-s}D^{\frac{1}{2}}(Gv)$ and v still denotes the solution of (3.85), then

$$(v_T, \Gamma_{-s}(v_T))_{-s} = \int_0^T \|D^{\frac{1}{2}}(Gv)\|_{-s}^2 dt \ge C\int_0^T \|Gv\|_{-s+\frac{1}{2}}^2 dt \ge C\|v_T\|_{-s}^2,$$

so that $\Gamma_{-s} : H_0^{-s}(\mathbb{T}) \to H_0^{-s}(\mathbb{T})$ is onto. The same is true for the map $v_T \in H_0^{-s}(\mathbb{T}) \to u(T) \in H_0^s(\mathbb{T})$. To prove (3.93), we establish first a weaker estimate

$$\|v_T\|_{-s}^2 \le C\left(\int_0^T \|Gv\|_{-s+\frac{1}{2}}^2 dt + \|v_T\|_{-s-\frac{1}{2}}^2\right). \tag{3.94}$$

We argue again by contradiction. If (3.94) is false, then there is a sequence (v_T^n) in $H_0^{-s}(\mathbb{T})$ such that

$$1 = \|v_T^n\|_{-s}^2 > n\left(\int_0^T \|Gv^n\|_{-s+\frac{1}{2}}^2 dt + \|v_T^n\|_{-s-\frac{1}{2}}^2\right). \tag{3.95}$$

It follows that

$$v_T^n \quad \to 0 \quad \text{in } H_0^{-s-\frac{1}{2}}(\mathbb{T}), \tag{3.96}$$

$$v^n \quad \to 0 \quad \text{in } C([0,T], H_0^{-s-\frac{1}{2}}(\mathbb{T})). \tag{3.97}$$

Let $w^n = (1-\partial_x^2)^{-\frac{s}{2}}v^n$. Then w^n solves

$$-w_t^n - \mathcal{H}w_{xx}^n + GDGw^n = (1-\partial_x^2)^{-\frac{s}{2}}[(1-\partial_x^2)^{\frac{s}{2}}, GDG]w^n = Bw^n,$$
$$w^n(T) = (1-\partial_x^2)^{-\frac{s}{2}}v_T^n =: w_T^n.$$

Then $\|w_T^n\|=1$, $w_T^n\to 0$ in $H_0^{-\frac12}(\mathbb{T})$, and

$$w^n \to 0 \quad \text{in } C([0,T],H_0^{-\frac12}(\mathbb{T})) \tag{3.98}$$

$$\int_0^T \|Gw^n\|_{\frac12}^2 dt \to 0. \tag{3.99}$$

For (3.99), we notice that

$$\begin{aligned}\int_0^T \|Gw^n\|_{\frac12}^2 dt &= \int_0^T \|G(1-\partial_x^2)^{-\frac{s}{2}}v^n\|_{\frac12}^2 dt\\ &\le \int_0^T \|Gv^n\|_{-s+\frac12}^2 dt + \int_0^T \|[G,(1-\partial_x^2)^{-\frac{s}{2}}]v^n\|_{\frac12}^2 dt. \end{aligned}\tag{3.100}$$

The first term in the right hand side of (3.100) tends to 0 by (3.95). For the second one, we have that

$$\int_0^T \|[G,(1-\partial_x^2)^{-\frac{s}{2}}]v^n\|_{\frac12}^2 dt \le C\int_0^T \|v^n\|_{-s-\frac12}^2 dt \le C\|v^n\|^2_{L^\infty(0,T,H^{-s-\frac12}(\mathbb{T}))} \to 0,$$

by (3.97).

From (3.91), we infer that

$$\|w^n\|_{L^2(0,T,H^{\frac12}(\mathbb{T}))} \le C\|w_T^n\|. \tag{3.101}$$

Arguing as in Step 1 and using (3.101), we can derive the following observability inequality

$$\|w_T^n\|^2 \le C\left(\int_0^T\int_{\mathbb{T}} |D^{\frac12}(Gw^n)|^2 dxdt + \|Bw^n\|^2_{L^2(0,T,H^{-\frac12}(\mathbb{T}))}\right).$$

Combined with (3.98) and (3.99), this yields $w_T^n\to 0$ in $H_0^0(\mathbb{T})$, contradicting the fact that $\|w_T^n\|=1$ for all n. The proof of (3.94) is complete. Finally, we prove (3.93) by contradiction. If (3.93) is false, then there is a sequence (v_T^n) in $H_0^{-s}(\mathbb{T})$ such that

$$1=\|v_T^n\|_{-s}^2 > n\int_0^T \|Gv^n\|_{-s+\frac12}^2 dt. \tag{3.102}$$

Extracting a subsequence still denoted by (v_T^n), we can assume that (v_T^n) is strongly convergent in $H_0^{-s-\frac12}(\mathbb{T})$ by compactness of the embedding $H_0^{-s}(\mathbb{T})\subset H_0^{-s-\frac12}(\mathbb{T})$. Using (3.102), we infer from (3.94) that (v_T^n) is also strongly convergent in $H_0^{-s}(\mathbb{T})$. Its limit v_T satisfies $\|v_T\|_{-s}=1$, and the solution v of (3.85) satisfies $Gv=0$ by (3.102). Thus for a.e. $t\in(0,T)$

$$v_{xxx}(\cdot,t)=\mathcal{H}v_{xxx}(\cdot,t)=0 \qquad \text{on } \omega.$$

We conclude with Lemma 2.9 that $v\equiv 0$, hence $v_T=0$, which contradicts $\|v_T\|_{-s}=1$. The proof of (3.93) is achieved.

Step 3. Fixed-point argument in $H_0^s(\mathbb{T})$.
We proceed as in [36]. Pick any $s\in(\frac12,2]$ and any $T>0$. We still denote by $(S(t))_{t\ge0}$ the

semigroup introduced in Lemma 2.15 and by $Z_{s,T}$ the space introduced in (2.58). For $v \in Z_{s,T}$, we set

$$\omega(v) = \int_0^T S(T-t)(vv_x)(t)\,dt.$$

From Step 2 we know that the linearized system, namely (3.84), with initial data $u_0 \in H_0^s(\mathbb{T})$ and control function $k \in L^2(0,T,H_0^s(\mathbb{T}))$ is well-posed and exactly controllable in $H_0^s(\mathbb{T})$. By a classical functional analysis argument (see e.g. [8, Lemma 2.48 p. 58]), one can construct a continuous operator $\Lambda : H_0^s(\mathbb{T}) \to L^2(0,T,H_0^s(\mathbb{T}))$ such that for any $u_1 \in H_0^s(\mathbb{T})$ the solution u of (3.84) associated with $u_0 = 0$ and $k = \Lambda(u_1)$ satisfies $u(T) = u_1$. Let us denote by $u = W(k)$ the corresponding trajectory. We know from Proposition 2.16 that W is continuous from $L^2(0,T,H_0^s(\mathbb{T}))$ into $Z_{s,T}$. Let $u_0, u_1 \in H_0^s(\mathbb{T})$ be given with

$$\|u_0\|_{H_0^s(\mathbb{T})} < \delta, \qquad \|u_1\|_{H_0^s(\mathbb{T})} < \delta,$$

where $\delta > 0$ will be chosen later. Let $v \in Z_{s,T}$. If we choose $k = \Lambda(u_1 - S(T)u_0 + \omega(v))$, then

$$S(t)u_0 - \int_0^t S(t-\tau)(vv_x)(\tau)d\tau + W(k)(t) = \begin{cases} u_0 & \text{if } t = 0; \\ u_1 & \text{if } t = T. \end{cases}$$

It suggests to consider the nonlinear map $v \to \Gamma(v)$, where

$$\Gamma(v)(t) = S(t)u_0 - \int_0^t S(t-\tau)(vv_x)(\tau)d\tau + W(\Lambda(u_1 - S(T)u_0 + \omega(v)))(t).$$

The proof will be complete if we can show that this map has a fixed point in the space $Z_{s,T}$. Using the estimates in the proof of Theorem 2.13, we see that

$$\|\omega(v)\|_s \le C\|\int_0^t S(t-\tau)(vv_x)(\tau)d\tau\|_{Z_{s,T}} \le C\|v\|^2_{Z_{s,T}}$$

and that there are some constants $C_0 > 0$ and $C_1 > 0$ such that

$$\begin{aligned} \|\Gamma(v)\|_{Z_{s,T}} &\le C_0(\|u_0\|_s + \|u_1\|_s) + C_1\|v\|^2_{Z_{s,T}} && \forall v \in Z_{s,T}, \\ \|\Gamma(v^1) - \Gamma(v^2)\|_{Z_{s,T}} &\le C_1\big(\|v^1\|_{Z_{s,T}} + \|v^2\|_{Z_{s,T}}\big)\|v^1 - v^2\|_{Z_{s,T}} && \forall v^1, v^2 \in Z_{s,T}. \end{aligned}$$

Let $B = \{v \in Z_{s,T};\ \|v\|_{Z_{s,T}} \le R\}$. We choose the radius R in such a way that the ball B is left invariant by Γ and Γ contracts in B, i.e.

$$C_0(\|u_0\|_s + \|u_1\|_s) + C_1R^2 \le R, \quad \text{and } 2C_1R < 1.$$

It is sufficient to take $R = (4C_1)^{-1}$ and $\delta := R/(4C_0)$. The proof of Theorem 3.1 is complete. □

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